香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

FINAL EXAMINATION

Course Code: MATH 4033
Course Title: Calculus on Manifolds
Semester: Spring 2020-21
Date and Time: 12:30PM-3:30PM, 21 May 2019

Instructions

- No discussion with any person, online or offline, is allowed. Posting anything in social media during the test is not allowed.
- It is an **OPEN-NOTES** exam. You can view the instructor's lecture notes, homework solutions, your own notebooks.
- You can search the internet for reference if needed. However, you should not open any social media site or online chatting apps.
- Answer **ALL** problems. Write your solutions on your own answer sheets, and scan it as a PDF.
- You must **SHOW YOUR WORK** and **JUSTIFY YOUR STEPS** to receive credits in every problem in Part B.
- Some problems in Part B are structured into several parts. You can quote the results stated in the preceding parts to do the next part.
- On top of the first page of your answer sheets, write down:

> "I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."
>
> YOUR SIGNATURE

- Sign on the **top right corner of EVERY page** of your answer sheets.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

Part A - Short Questions (25 points)

[Recommended time: $<$ 30 min.]
Instruction: Write your answers your own answer sheets.

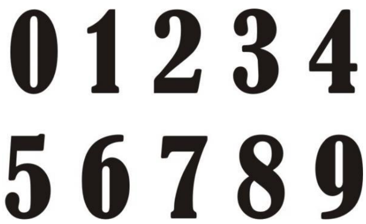
1. Which ONE of the following mathematicians was never mentioned in the course? **[2]**
   - A. Georges de Rham
   - B. George Stokes
   - C. Henri Poincaré
   - D. Srinivasa Ramanujan
   - E. Élie Cartan

2. Is it always true that $\omega \wedge \omega = 0$ for any differential form ω in $\mathbb{R}^4$? If true, explain briefly why. If false, give a simple counter-example. **[3]**

3. Is it always true that: **[3]**
$$du^1 \wedge du^2 = \det \frac{\partial(u_1, u_2)}{\partial(v_1, v_2)} dv^1 \wedge dv^2$$
   where (u_1, u_2, u_3) and (v_1, v_2, v_3) are overlapping local coordinates of $\mathbb{R}^3$? If true, explain briefly why. If false, give a simple counter-example.

4. Suppose V and W are two finite-dimensional vector spaces such that the following is an exact sequence: **[4]**
$$0 \to V \to W \to V \to 0.$$
   Which of the following must be true? List ALL correct answer(s):
   - A. $\dim V$ is even
   - B. $\dim W$ is even
   - C. $\dim V \leq \dim W$
   - D. $\dim W \leq \dim V$

5. Consider the ten digits and view them as an open (i.e. no boundary) solid region in $\mathbb{R}^2$:



   (a) Using a Mayer-Vietoris sequence, determine H^k_{dR} of the digit 8 for $k = 0, 1, 2$. **[5]**

   (b) Answer each question below – you can write "NONE". No justification is needed. **[4]**
   - i. List ALL digit(s) that has/have trivial H^0_{dR}.
   - ii. List ALL digit(s) that has/have trivial H^1_{dR}.
   - iii. List ALL digit(s) that has/have trivial H^2_{dR}.
   - iv. List ALL digit(s) that can deformation retract onto the digit 0.

6. Let X be a vector field on a C^∞ manifold M. Given a short proof that when acting on differential forms on M, we have $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$. **[4]**

Part B - Long Questions (75 points): Answer ALL THREE problems

[Recommended time: Q1 $<$ 45 min; Q2 $<$ 1hr; Q3 $<$ 45 min]
Instruction: Write your solutions in your own answer sheets.

---

1. Let M be the following subset of $\mathbb{R}^2 \times \mathbb{RP}^1$:

$$M := \{\big((x_1, x_2), [y_1 : y_2]\big) \in \mathbb{R}^2 \times \mathbb{RP}^1 : x_1 y_2 = x_2 y_1\}.$$

   (a) Show that M is a C^∞ manifold by verifying that (i) the following local parametrizations cover the whole M, and (ii) their transition maps are smooth on their domains (please specify their domains too). **[8]**

$$F(u_1, u_2) = \big((u_1, u_1 u_2), [1 : u_2]\big) : \mathbb{R}^2 \to M$$
$$G(v_1, v_2) = \big((v_1 v_2, v_1), [v_2 : 1]\big) : \mathbb{R}^2 \to M$$

   (b) Determine whether or not M is orientable. Justify your answer. **[6]**

   (c) Let Σ be a subset of M defined by: **[8]**

$$\Sigma := \left\{\big((0, 0), [y_1 : y_2]\big) : [y_1 : y_2] \in \mathbb{RP}^1\right\}.$$

   Show that Σ is a deformation retract of M. Construct the Ψ_t map explicitly, and verify all conditions (including why Σ is a submanifold of M, and why each Ψ_t is smooth, etc.)

2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function such that $f^{-1}((-\infty, 1])$ is compact and has non-zero volume. It is also given that both $f^{-1}(0)$ and $f^{-1}(1)$ are non-empty, and $\nabla f \neq 0$ on both $f^{-1}(0)$ and $f^{-1}(1)$.

   (a) Cite the number of the theorem/proposition/lemma/corollary (e.g. Theorem 7.77) in the instructor's lecture notes that shows each result below: **[4]**
      i. $f^{-1}((-\infty, 1])$ is a manifold with boundary $f^{-1}(1)$.
      ii. $f^{-1}(0)$ and $f^{-1}(1)$ are $(n-1)$-dimensional submanifolds of $\mathbb{R}^n$

   (b) Let $M := \mathbb{R}^n \backslash f^{-1}(0)$. Show that $f^{-1}(1)$ is a submanifold of M. **[3]**

   (c) Denote the standard coordinates of $\mathbb{R}^n$ by $(x_1, \cdots, x_n)$. Consider the $(n-1)$-form defined on M: **[8]**

$$\omega := \sum_{i=1}^{n} \frac{(-1)^{i-1} x_i}{f(x_1, \cdots, x_n)} \, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n.$$

   Show that ω is closed on $M := \mathbb{R}^n \backslash f^{-1}(0)$ if and only if $nf(x_1, \cdots, x_n) = \langle X, \nabla f \rangle$ on $\mathbb{R}^n \backslash f^{-1}(0)$, where $X = (x_1, \cdots, x_n)$ is the position vector in $\mathbb{R}^n$.

   (d) Denote by $\iota : f^{-1}(1) \to M$ the inclusion map. By calculating **[12]**

$$\int_{f^{-1}(1)} \iota^* \omega,$$

   show that ω is not exact on M.

3. Given a compact, orientable smooth manifold M with $m := \dim M \geq 2$ and with non-empty boundary ∂M, we construct $\widehat{M}$ with a pair of M's gluing along their boundaries, i.e.
$$\widehat{M} = (M \times \{0,1\})/\sim$$
where the equivalence relation $\sim$ is defined by: for any $p, q \in M$ and $m, n \in \{0,1\}$, we declare that:
$$(p,n) \sim (q,m) \Longleftrightarrow p,q \in \partial M \text{ and } p = q$$
For example, if M is the 2-dimensional hemisphere , then $\widehat{M}$ is the whole sphere. It can be shown that $\widehat{M}$ is an orientable smooth manifold whenever M is so. We denote:
$$s(M) := \sum_{k=0}^{\infty} (-1)^k \dim H^k_{\text{dR}}(M)$$
and similarly for $s(\widehat{M})$ and $s(\partial M)$.

(a) Explain why $s(M)$ is in fact a finite sum. **[2]**

(b) Using Mayer-Vietoris sequence(s), show that **[14]**
$$s(\widehat{M}) - 2s(M) + s(\partial M) = 0.$$
Justify every step with at least a short reason, with the help of pictures if needed.

(c) Using the following fact without proof: **[10]**

"For any compact, orientable smooth n-manifold N without boundary, we have $\dim H^k_{\text{dR}}(N) = \dim H^{n-k}_{\text{dR}}(N)$ for any $k \in \{0, 1, \cdots, n\}$",

show that $s(\partial M)$ must be an even number.

---

* End of Paper *