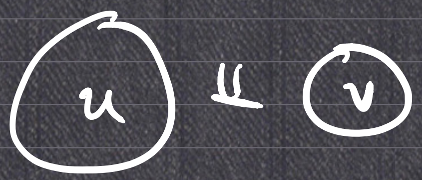
$$H^k_{dR}(M) = \mathrm{Ker}(d : \Lambda^k \to \Lambda^{k+1}) \Big/ \mathrm{Im}(d : \Lambda^{k-1} \to \Lambda^k) \qquad \underline{k \geq 1}.$$

$$H^0_{dR}(M) := \mathrm{Ker}(d : \Lambda^0 \to \Lambda^1) \big/ \{0\}$$

$$= \mathrm{Ker}(d : \underset{\substack{\uparrow \\ \text{functions}}}{\Lambda^0} \to \Lambda^1) = \{f \in C^\infty(M) : \underbrace{df = 0}\}.$$

$$= \left\{ \sum_{i=1}^{N} c_i \mathbb{1}_{U_i} : c_i \in \mathbb{R} \right\}.$$

$$= \mathrm{span}\{\mathbb{1}_{U_i}\}_{i=1}^{N}$$

$M =$ 

$$M = \coprod_{i=1}^{N} U_i$$

$\dim H^0_{dR}(M) = \#$ connected components of M.

$M = \underset{\text{open}}{U \cup V}$

$$\begin{tikzcd}
 & U \arrow[dl, "j_U"'] & \\
U \cup V & & U \cap V \arrow[ul, "i_U"'] \arrow[dl, "i_V"] \\
 & V \arrow[ul, "j_V"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & \Lambda^k(U) \arrow[dr, "i_U^*"] & \\
\Lambda^k(U \cup V) \arrow[ur, "j_U^*"] \arrow[dr, "j_V^*"'] & & \Lambda^k(U \cap V) \\
 & \Lambda^k(V) \arrow[ur, "i_V^*"'] &
\end{tikzcd}$$

Claim:

$$\begin{array}{ccccccccc}
 & & \omega & \longmapsto & (j_U^*\omega, j_V^*\omega) & & & & \\
0 & \longrightarrow & \Lambda^k(U \cup V) & \xrightarrow{j_U^* \oplus j_V^*} & \Lambda^k(U) \oplus \Lambda^k(V) & \xrightarrow{i_U^* - i_V^*} & \Lambda^k(U \cap V) & \longrightarrow & 0 \\
 & & & & (\alpha, \beta) & \longmapsto & i_U^*\alpha - i_V^*\beta & &
\end{array}$$

is exact.

$$\omega = \sum_i \omega_i \, d\underset{\sim}{x^i} \quad \text{on } U \cup V$$

parametrization on $U \cup V$.

$$j_U^* \omega = \underbrace{\sum_i \omega_i \circ j_U \, dx^i}_{\omega|_U} \quad \text{on } U.$$

$$I \xrightarrow{\iota} \quad f : M \to \mathbb{R}, \quad f \circ \iota : I \to \mathbb{R}.$$

- $j_U^* \oplus j_V^*$ is injective.

<u>Proof</u>: If $(j_U^* \oplus j_V^*)(\omega) = 0$,

then $j_U^* \omega = 0$ and $j_V^* \omega = 0$

$\Rightarrow \omega|_U = 0$ and $\omega|_V = 0$.

$\Rightarrow \omega = 0$ on $U \cup V$.

$\therefore \operatorname{Ker}(j_U^* \oplus j_V^*) = \{0\}$ ∎

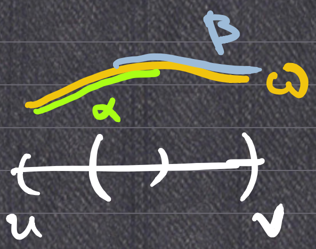
- $\operatorname{Im}(j_U^* \oplus j_V^*) \subset \operatorname{Ker}(i_U^* - i_V^*)$

<u>Proof</u>: let $(\alpha, \beta) \in \operatorname{Im}(j_U^* \oplus j_V^*)$.

$\Rightarrow (\alpha, \beta) = (j_U^* \oplus j_V^*)(\omega) \quad \exists \omega \in \Lambda^k(U \cup V)$

$= (\underline{j_U^* \omega}, \underset{\sim}{j_V^* \omega})$

$\Rightarrow \omega|_U = \alpha, \quad \omega|_V = \beta.$



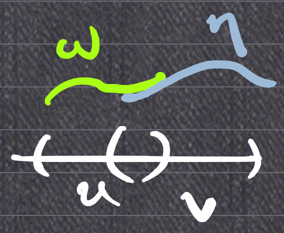
$$(i_U^* - i_V^*)(\alpha, \beta) = i_U^* \alpha - i_V^* \beta$$
$$= \alpha|_{U \cap V} - \beta|_{U \cap V}$$
$$= \omega|_{U \cap V} - \omega|_{U \cap V}$$
$$= 0.$$

$\therefore (\alpha,\beta) \in \mathrm{Ker}(i_U^* - i_V^*)$

∎

- $\mathrm{Ker}(i_U^* - i_V^*) \subset \mathrm{Im}(j_U^* \oplus j_V^*)$

<u>Proof</u>: $(\omega,\eta) \in \mathrm{Ker}(i_U^* - i_V^*)$



$\Rightarrow \quad i_U^*\omega - i_V^*\eta = 0$

$\Rightarrow \quad \omega|_{U\cap V} = \eta|_{U\cap V}$

let $\sigma = \begin{cases} \omega & \text{on } U \\ \eta & \text{on } V. \end{cases}$ (well-defined on $U\cap V$)

$\sigma|_U = \omega, \quad \sigma|_V = \eta$

$\Rightarrow \quad j_U^*\sigma = \omega, \quad j_V^*\sigma = \eta$

$\Rightarrow \quad (\omega,\eta) = (j_U^* \oplus j_V^*)(\sigma) \in \mathrm{Im}(j_U^* \oplus j_V^*).$

∎

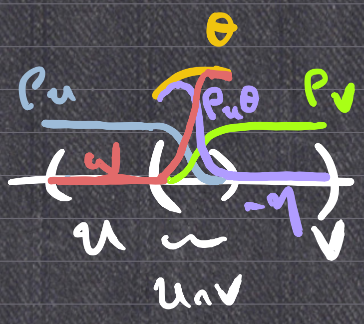
- $i_U^* - i_V^*$ is surjective.

<u>Proof</u>: Pick $\theta \in \Lambda^k(U\cap V)$

<u>WANT</u>: $(i_U^* - i_V^*)(\omega,\eta) = \theta.$ on U on V

find then

$\therefore$ find $\omega|_{U\cap V} - \eta|_{U\cap V} = \theta.$



$\{\rho_U, \rho_V\}$ partition of unity

on V $\quad \eta = \begin{cases} -\rho_U\theta & \text{on } U\cap V \\ 0 & \text{on } V\setminus U \end{cases}$

on U $$\omega = \begin{cases} \rho_V \theta & \text{on } U\cap V \\ 0 & \text{on } U\setminus V \end{cases}$$

$$(i_U^* - i_V^*)(\omega,\eta) = i_U^*\omega - i_V^*\eta = \omega|_{U\cap V} - \eta|_{U\cap V}$$
$$= \rho_V\theta - (-\rho_U\theta)$$
$$= \underbrace{(\rho_U + \rho_V)}_{1}\theta = \theta.$$

$\boxtimes$

$$0 \longrightarrow \Lambda^k(U\cup V) \xrightarrow{j_U^* \oplus j_V^*} \Lambda^k(U) \oplus \Lambda^k(V) \xrightarrow{i_U^* - i_V^*} \Lambda^k(U\cap V) \longrightarrow 0$$

induces maps on:

$$(*) \quad 0 \longrightarrow H^k_{dR}(U\cup V) \xrightarrow{j_U^* \oplus j_V^*} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{i_U^* - i_V^*} H^k_{dR}(U\cap V) \to 0$$
$$[\omega] \longmapsto ([j_U^*\omega], [j_V^*\omega])$$

$(*)$ may not be exact:

Counterexample

$M = \mathbb{R}^2 \setminus \{(0,0)\} = U \cup V.$

$U = \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}.$

$V = \mathbb{R}^2 \setminus \{(x,0) : x \leq 0\}.$

$U \cap V =$ 

$$H^1_{dR}(U\cup V) = H^1_{dR}(\mathbb{R}^2 \setminus \{(0,0)\}) \neq 0$$

$H^1_{dR}(U) = \cdots = 0$

Star-shape.

$H^1_{dR}(V) = 0.$

(Zigzag's lemma)

Given short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \to & \Lambda^0(U\cup V) & \to & \Lambda^0(U)\oplus\Lambda^0(V) & \to & \Lambda^0(U\cap V) & \to & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \to & \Lambda^1(U\cup V) & \to & \Lambda^1(U)\oplus\Lambda^1(V) & \to & \Lambda^1(U\cap V) & \to & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \to & \Lambda^2(U\cup V) & \to & \Lambda^2(U)\oplus\Lambda^2(V) & \to & \Lambda^2(U\cap V) & \to & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

induce a loooooooong exact sequence:

$$
\begin{tikzcd}
0 \arrow[r] & H^0(U\cup V) \arrow[r] & H^0(U)\oplus H^0(V) \arrow[r] & H^0(U\cap V) \arrow[dll] \\
 & H^1(U\cup V) \arrow[r] & H^1(U)\oplus H^1(V) \arrow[r] & H^1(U\cap V) \arrow[dll] \\
 & H^2(U\cup V) \arrow[r] & H^2(U)\oplus H^2(V) \arrow[r] & H^2(U\cap V) \arrow[dll] \\
 & H^3(U\cup V) \arrow[r] & H^3(U)\oplus H^3(V) \arrow[r] & H^3(U\cap V) \arrow[d] \\
 & & & \;
\end{tikzcd}
$$

Mayer–Vietoris sequence.

First isomorphism theorem.

$T: G \to H$ homomorphism between groups G, H.

then $G/\ker T \cong \mathrm{Im}(T)$

<u>Proof</u>: $\Phi : G/\ker T \longrightarrow \operatorname{Im}(T)$

$$[g] \mapsto T(g) \qquad \blacksquare$$

$$\dim\left(V/\ker T\right) = \dim \operatorname{Im}(T) \qquad T: V \to W$$

$$\dim V - \dim \ker T = \dim \operatorname{Im}(T)$$

$$\dim V = \underbrace{\dim \ker T}_{\text{nullity}} + \underbrace{\dim \operatorname{Im}(T)}_{\text{rank.}}$$

<u>Lemma</u>: Given an exact sequence of vector spaces:

$$0 \xrightarrow{T_0} V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \xrightarrow{T_3} \cdots \longrightarrow V_{n-1} \xrightarrow{T_{n-1}} V_n \xrightarrow{T_n} 0$$

then

$$\dim V_1 - \dim V_2 + \dim V_3 - \cdots + (-1)^{n-1} \dim V_n = 0.$$

<u>Proof</u>: $\underbrace{\dim \ker T_n}_{\overset{\shortparallel}{\dim V_n}} = \dim \operatorname{Im} T_{n-1}$

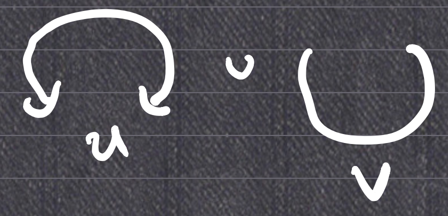
$$= \dim V_{n-1} - \dim \ker T_{n-1}$$

$$= \dim V_{n-1} - \dim \operatorname{Im} T_{n-2}$$

$$= \dim V_{n-1} - \left(\dim V_{n-2} - \dim \ker T_{n-2}\right)$$

$$= \cdots$$

$\blacksquare$

<u>e.g.</u> $M = S^1 =$ 

$U \cap V = (\quad \sqcup \quad)$

$$0 \longrightarrow \overset{1}{H^0(U \cup V)} \to \overset{2}{H^0(U) \oplus H^0(V)} \to \overset{2}{H^0(U \cap V)}$$

$$\hookrightarrow \overset{x}{H^1(U \cup V)} \to \overset{0}{H^1(U) \oplus H^1(V)} \to H^1(U \cap V)$$

$$\hookrightarrow 0$$

$U \to \cdot$

$V \to \cdot$

$$1 - 2 + 2 - x = 0 \Rightarrow x = 1$$

$$\therefore \dim H^1(S^1) = 1.$$

$$\therefore \dim H^1(\mathbb{R}^2 \setminus \{(0,0)\}) = 1.$$

$$\Rightarrow H^1(\mathbb{R}^2 \setminus 0) = \operatorname{span}\left\{\left[\frac{-y\,dx + x\,dy}{x^2+y^2}\right]\right\}$$

<u>e.g.</u> $U \cap V = \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$. distinct

where $U = \mathbb{R}^2 \setminus \{p_1, \ldots, p_{n-1}\}$.

$V = \mathbb{R}^2 \setminus \{p_n\}$.

$U \cup V = \mathbb{R}^2$

$$0 \longrightarrow \overset{1}{H^0(U \cup V)} \to \overset{2}{H^0(U) \oplus H^0(V)} \to \overset{1}{H^0(U \cap V)}$$

$$\hookrightarrow \overset{0}{H^1(U \cup V)} \to \overset{a_{n-1}+1}{H^1(U) \oplus H^1(V)} \to \overset{a_n}{H^1(U \cap V)}$$

$$\hookrightarrow \overset{0}{H^2(U \cup V)} \to H^2(U) \oplus H^2(V) \to H^2(U \cap V)$$

$$\hookrightarrow H^3(U \cup V) \to H^3(U) \oplus H^3(V) \to H^3(U \cap V)$$

$$1 - 2 + 1 - 0 - (a_{n-1} + 1) + a_n = 0 \Rightarrow \boxed{a_n = a_{n-1} + 1}$$

$a_1 = 1 \implies a_n = n.$

$\dim H^1_{dR}(\mathbb{R}^2 \setminus n \text{ pts}) = n.$