MATH 4033 • Spring 2021 • Calculus on Manifolds
Problem Set #4 • de Rham Cohomology • Due Date: Optional

1. The purpose of this exercise is to prove that $H^2(\mathbb{R}^3) = 0$, i.e. every closed 2-form on $\mathbb{R}^3$ must be exact. Consider a closed form:

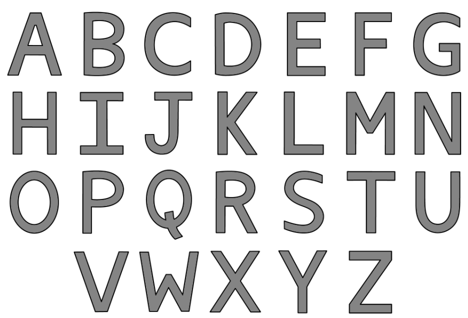
$$\omega = A\,dy \wedge dz + B\,dz \wedge dx + C\,dx \wedge dy$$

where A, B and C are smooth scalar functions of (x, y, z). Define the following 1-form:

$$\begin{aligned}\alpha := &\left(\int_0^1 A(tx, ty, tz)t\,dt\right)(y\,dz - z\,dy)\\ &+\left(\int_0^1 B(tx, ty, tz)t\,dt\right)(z\,dx - x\,dz)\\ &+\left(\int_0^1 C(tx, ty, tz)t\,dt\right)(x\,dy - y\,dx)\end{aligned}$$

First, compute $d\alpha$; then use the result to show that ω is exact.

2. Consider the following alphabet. Each letter is regarded as a solid region.



Answer the following without justification:

(a) Which letter(s) is/are contractible?

(b) Which letter(s) is/are star-shaped?

(c) Which letter(s) has/have non-zero 1st Betti number b_1?

3. Prove the following statements about deformation retracts by explicitly constructing Ψ_t.

(a) Show that the Möbius strip Σ defined in Example 4.11 deformation retracts onto a circle. [Hence, $H^1_{\text{dR}}(\Sigma) = H^1_{\text{dR}}(\mathbb{S}^1) = \mathbb{R}$.]

(b) The zero section Σ_0 of the tangent bundle TM of a smooth manifold M is defined to be:

$$\Sigma_0 := \{(p, 0_p) \in p \times T_pM : p \in M\}$$

where 0_p is the zero vector in T_pM. Show that Σ_0 is a deformation retract of TM. [Hence, $H^*_{\text{dR}}(TM) = H^*_{\text{dR}}(\Sigma_0) = H^*_{\text{dR}}(M)$.]

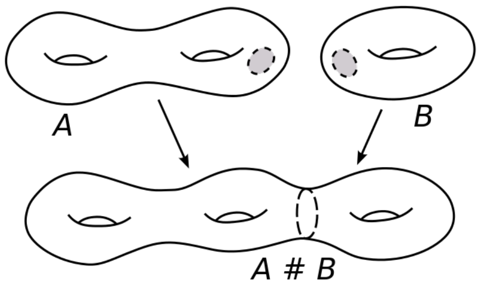
In the following problems, you may assume the Poincaré's Lemma and Deformation Retract Invariance hold on any H^k. Also, we may use the following fact without proof:

On a compact, connected orientable manifold M without boundary, then:

- $\dim H^n(M) = 1$ where $n = \dim M$
- $H^n(M\backslash\{p\}) = 0$ for any $p \in M$.

4. Let $\mathbb{T}^2$ be the 2-dimensional torus. Show that $b_1(\mathbb{T}^2) = 2$.

5. Given two compact smooth 2-manifolds A and B without boundary, its connected sum $A\#B$ is a 2-manifold obtained by removing an open ball in each of A and B, and then gluing them along the two boundary circles:



   (a) Show that $A\#B$ is orientable if both A and B are so. [Hint: use partitions of unity to construct a global non-vanishing 2-form.]
   (b) Using Mayer-Vietoris sequence, show that $b_1(A\#B) = b_1(A) + b_1(B)$.

6. (∞ points (bonus)) Prove or disprove: "Every Hodge cohomology class of a non-singular complex projective manifold $X \subset \mathbb{CP}^N$ is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of X."

End of all MATH 4033 homework.