§5 - de Rham cohomology

$\beta = d\alpha \Rightarrow d\beta = d(d\alpha) = 0.$

(exact $\Rightarrow$ closed)

closed $\not\Rightarrow$ exact.

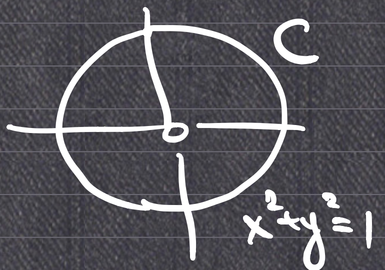
"How many" closed forms are not exact?

e.g. $M = \mathbb{R}^2 \setminus \{(0,0)\}$.

$$\alpha := -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \quad \text{(defined on } M\text{)}$$

$d\alpha = 0$ (exercise)

$\iota : C \to M$



$$\int_C \iota^*\alpha = \int_C -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= \int_0^{2\pi} -\frac{\sin t}{1}d(\cos t) + \frac{\cos t}{1}d(\sin t) \quad (x,y) = (\cos t, \sin t)$$

$$= 2\pi.$$

If $\underline{\alpha = df}$, then $\iota^*\alpha = \iota^* df = d(\underset{\sim}{\iota^* f})$ for

$$\int_C \iota^*\alpha = \int_C \underbrace{d(\iota^* f)} = \int_{\partial C} \iota^* f = 0. \qquad \partial C = \phi$$

<u>Contradiction!</u> $\therefore\ \alpha$ is not exact.

Higher dimension:

$M = \mathbb{R}^{n+1} \setminus \{\vec{0}\}$. $\quad \vec{x} = (x_1, \ldots, x_{n+1})$

$$\alpha := \sum_{j=1}^{n+1} (-1)^j \frac{x_j}{|\vec{x}|^2} dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{n+1}$$

n-form.

HW Check $d\alpha = 0$.

but not exact

If $d\alpha = 0$, but α is not exact.

then $\alpha + d\beta$ is another such example.

$$d(\alpha + d\beta) = d\alpha + d^2\beta = 0$$

$c\alpha$ is another such example.

$\uparrow$

constant

$$\ker(d : \Lambda^k T^*M \to \Lambda^{k+1} T^*M) = \{\alpha \in \Lambda^k T^*M : d\alpha = 0\}.$$

= set of all closed k-forms on M

$$\operatorname{Im}(d : \Lambda^{k-1} T^*M \to \Lambda^k T^*M) = \{d\beta : \beta \in \Lambda^{k-1} T^*M\}$$

= set of all exact k-forms on M.

$$H^k_{dR}(M) := \frac{\mathrm{Ker}(d : \Lambda^k T^*M \to \Lambda^{k+1} T^*M)}{\mathrm{Im}(d : \Lambda^{k-1} T^*M \to \Lambda^k T^*M)}$$

kth de Rham cohomology group of M.

$[\omega] \in H^k_{dR}(M)$ $\qquad [\omega] = [\omega + d\beta]$.

ω k-form on M

$d\omega = 0$

Theorem If M and N are diffeomorphic, then $H^k_{dR}(M)$ and $H^k_{dR}(N)$ are isomorphic (as vector spaces).

Proof: Let $\Phi : M \to N$ be a diffeomorphism.

$$\Phi^* : \Lambda^k T^*N \to \Lambda^k T^*M.$$

$$\tilde{\Phi}^* : H^k_{dR}(N) \to H^k_{dR}(M)$$

Claim: $\tilde{\Phi}^*(\underbrace{[\omega]}_{\in H^k_{dR}(N)}) := \underbrace{[\Phi^*\omega]}_{\in H^k_{dR}(M)}$ is well-defined.

Proof: $d\omega = 0 \Rightarrow d(\underbrace{\Phi^*\omega}_{\therefore\ \text{closed}}) = \Phi^*(d\omega) = \Phi^*(0) = 0$.

If $[\omega] = [\omega']$, $\quad \omega - \omega' = d\beta$

$$\Phi^*\omega - \Phi^*\omega' = \Phi^*(d\beta) = d(\Phi^*\beta).$$

$$\Rightarrow [\Phi^*\omega] = [\Phi^*\omega'] \text{ in } H^k_{dR}(M).$$

$$\widetilde{(\Phi^{-1})^*} : H^k_{dR}(M) \to H^k_{dR}(N)$$

$$\widetilde{(\Phi^{-1})^*}([\eta]) = [(\Phi^{-1})^*\eta].$$

$$\widetilde{\Phi^*} \circ \widetilde{(\Phi^{-1})^*}([\eta]) = \widetilde{\Phi^*}([(\Phi^{-1})^*\eta])$$

$$= [\Phi^*(\Phi^{-1})^*\eta]$$

$$= [\underbrace{(\Phi^{-1}\circ\Phi)}_{id}{}^*\eta] = [\eta].$$

Similarly: $\widetilde{(\Phi^{-1})^*} \circ \widetilde{\Phi^*}([\omega]) = [\omega]$.

$\therefore\ \widetilde{\Phi^*} : H^k_{dR}(N) \to H^k_{dR}(M)$ is an isomorphism.

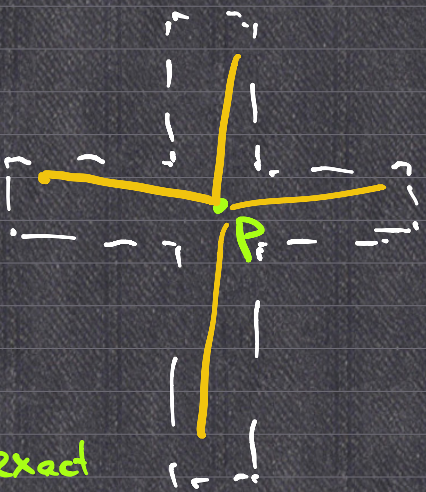
Notation:

$\Phi^*([\omega]) = [\Phi^*\omega]$

Poincaré Lemma.

If $\underset{\text{open}}{U \subset \mathbb{R}^n}$ star-shaped.

then $H^k_{dR}(U) = 0 \quad \forall k \geq 1$



Proof of case $k=1$

Want: any closed 1-form on U is exact

let ω be a closed 1-form on $\mathcal{U}$.

Guess: $f(\vec{x}) = \int_{L_{\vec{x}}} \iota^* \omega$

$\vec{V} = \nabla f$

$\uparrow$

potential function.

$\vec{r}_{\vec{x}}(t) = (1-t)\vec{p} + t\vec{x}$, $0 \le t \le 1$

let $\omega = \sum_{i=1}^{n} \omega_i(\vec{x})\, dx^i$,

$\iota^* dx^i = d((1-t)p_i + tx_i)$
$= (x_i - p_i)\, dt$

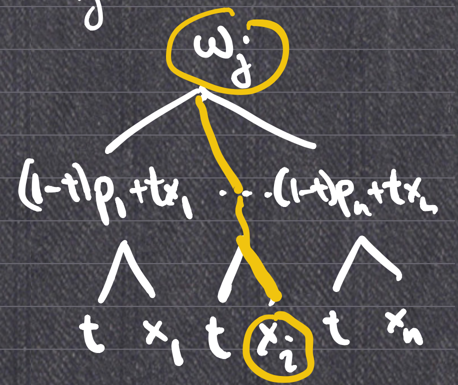
$$f(\vec{x}) = \int_{L_{\vec{x}}} \iota^* \omega = \int_{t=0}^{t=1} \sum_i \omega_i(\vec{r}_x(t)) \cdot (x_i - p_i)\, dt$$

$$\frac{\partial f}{\partial x_i} = \int_{t=0}^{t=1} \frac{\partial}{\partial x_i} \sum_{j=1}^{n} \omega_j(\vec{r}_x(t)) \cdot (x_j - p_j)\, dt$$

$$= \sum_{j=1}^{n} \int_{t=0}^{t=1} \frac{\partial}{\partial x_i}\omega_j(\vec{r}_x(t)) \cdot (x_j - p_j) + \omega_j(\vec{r}_x(t)) \cdot \delta_{ij}\, dt$$

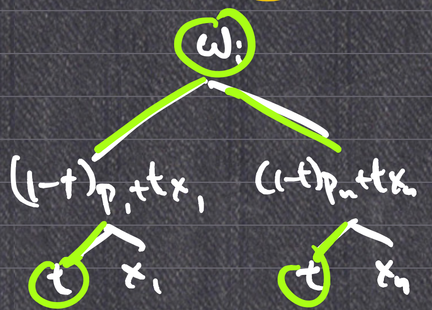
$$= \sum_{j=1}^{n} \int_{t=0}^{t=1} \frac{\partial \omega_j}{\partial x_i} t \cdot (x_j - p_j) + \omega_j(\vec{r}_x(t))\, \delta_{ij}\, dt$$

$$= \int_{t=0}^{t=1} \sum_{j=1}^{n} \frac{\partial \omega_j}{\partial x_i} t (x_j - p_j) + \omega_i(\vec{r}_x(t))\, dt$$



$$= \int_{t=0}^{t=1} \underbrace{\sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x_j} t (x_j - p_j)} + \omega_i(\vec{r}_x(t))\, dt$$

$$= \int_0^1 t \frac{\partial}{\partial t} \omega_i(\vec{r}_x(t)) + \omega_i(\vec{r}_x(t))\, dt$$



$$= \int_0^1 \frac{\partial}{\partial t}\left(t\, \omega_i(\vec{r}_x(t))\right) dt \qquad \frac{\partial}{\partial t}\omega_i(\vec{r}_x(t))$$

$$= \left[t\omega_i(\vec{r}_x(t))\right]_0^1 \qquad = \sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x_j}(x_j - p_j)$$

$$= \omega_i(\underbrace{\vec{r}_x(1)}_{\vec{x}}) - \cancel{0}.$$

$$= \omega_i(\vec{x}).$$

$$\therefore \boxed{df = \omega.}$$

$U \subset \mathbb{R}^2$ is simply-connected ($U \neq \mathbb{R}^2$)

$\xRightarrow{\text{RMT}}$ U diffeomorphic to

diffeo. inv.

$\Rightarrow$ $H^1_{dR}(U) = H^1_{dR}(\;) = 0$

star-shaped.

$\therefore$ All closed 1-forms on U are exact.

$\therefore$ All curl-zero vector field on U are conservative.

$$H^1_{dR}(\mathbb{R}^2 \setminus \{0\}) \neq 0.$$

$$\underbrace{\left[-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right]}_{\in H^1_{dR}(\mathbb{R}^2\setminus\{0\})} \neq [0]$$

$\therefore$ $\mathbb{R}^2 \setminus \{0\}$ is not diffeomorphic to any star-shaped open set in $\mathbb{R}^2$.