MATH 4033 • Spring 2021 • Calculus on Manifolds
Problem Set #3 • Stokes' Theorem • Due Date: 16/05/2019, 11:59PM

1. (15 points) (a) Show that any complex manifold is orientable. [Note: The way I computed the determinant of a block matrix in class was not generally correct. Try to fix it.]
   (b) Show that any symplectic manifold is orientable. A manifold M is said to be symplectic if there exists a closed 2-form ω on M such that whenever X is a tangent vector such that $i_X\omega = 0$, then $X = 0$.
   (c) Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞ function such that $f^{-1}(0) \neq \emptyset$, and $\nabla f(p) \neq 0$ for any $p \in f^{-1}(0)$. Show that $\Sigma := f^{-1}(0)$ is orientable.

2. (10 points) Let Ω be a non-vanishing n-form on a manifold M without boundary (hence M is orientable). Show, from the definition of integral on n-forms, that $\int_M \Omega \neq 0$. Hence, show that $H^n_{\mathrm{dR}}(M) \neq 0$.

3. (10 points) Let Σ^n be an orientable regular hypersurface in $\mathbb{R}^{n+1}$, and Ω be a $(n+1)$-dimensional submanifold in $\mathbb{R}^{n+1}$ such that $\partial\Omega = \Sigma$. Let μ be the n-form on Σ defined as in Q3 of the midterm this year. Using the results proved in the midterm and the generalized Stokes' Theorem, prove that for any C^∞ vector field Y on $\mathbb{R}^{n+1}$, we have

$$\int_\Omega \nabla \cdot Y \, dV = \int_\Sigma (Y \cdot \nu)\, \mu$$

where $dV = dx^1 \cdots dx^{n+1}$, and ν is the outward-pointing unit normal vector to Σ.

4. (20 points) Consider the following torus $\mathbb{T}^2$ in $\mathbb{R}^4$:

$$\mathbb{T}^2 := \{(x, y, z, w) \in \mathbb{R}^4 : x^2 + y^2 = 1 \quad \text{and} \quad z^2 + w^2 = 1\},$$

which can be locally parametrized by $F : (0, 2\pi) \times (0, 2\pi) \to \mathbb{T}^2$:

$$F(\theta_1, \theta_2) = (\cos\theta_1,\, \sin\theta_1,\, \cos\theta_2,\, \sin\theta_2)$$

Denote $\iota : \mathbb{T}^2 \to \mathbb{R}^4$ to be the inclusion map. Consider the following 1-form on $\mathbb{T}^2$:

$$\sigma := \iota^*\big(y^3\, dx - (x^3 - 3x)\, dy + (w^3 - 3w)\, dz - z^3\, dw\big).$$

   (a) Show that σ is closed.
   (b) Let $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$ parametrized by $G(t) = (\cos t, \sin t)$. Consider the map $\Phi : \mathbb{S}^1 \to \mathbb{T}^2$ given by:

$$\underbrace{(p, q)}_{\text{coordinates in } \mathbb{R}^2} \mapsto \underbrace{\big(p,\, q,\, (p-q)/\sqrt{2},\, (p+q)/\sqrt{2}\big)}_{\text{coordinates in } \mathbb{R}^4}.$$

   Express $\Phi^*\sigma$ in terms of dt.
   (c) Using (b), show that σ is not exact.

5. (20 points) Let ω be the n-form on $\mathbb{R}^{n+1}\backslash\{0\}$ defined by:

$$\omega = \frac{1}{|x|^{n+1}} \sum_{i=1}^{n+1} (-1)^{i-1}\, x_i\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{n+1}$$

where $x = (x_1, \ldots, x_{n+1})$ and $|x| = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$. Denote by $\mathbb{S}^n$ the unit n-sphere centered at 0.

   (a) Let $\iota : \mathbb{S}^n \to \mathbb{R}^{n+1}$ be the inclusion map. Show that $\int_{\mathbb{S}^n} \iota^*\omega \neq 0$.
   (b) Hence, show that ω is closed but is not exact on $\mathbb{R}^{n+1}\backslash\{0\}$.