M orientable manifold

Ω non-vanishing C^∞ n-form on M.

$\uparrow$ orientation

$$\int_M \omega := \sum_\alpha \int_{F_\alpha(U_\alpha)} \rho_\alpha \underbrace{f_\alpha du_\alpha^1 \wedge \cdots \wedge du_\alpha^n}_{\omega} = \sum_\alpha \frac{\Omega\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}\right)}{\left|\Omega\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_n}\right)\right|} \int_U \rho_\alpha f_\alpha du_\alpha^1 \cdots du_\alpha^n.$$

$\partial M \rightarrow$ Convention: use $i_\eta \Omega$ as the orientation. η = outward normal.

Stokes' Theorem

Let M^n be an orientable mfd, ω is an $(n-1)$-form on M s.t. $\operatorname{supp}\omega = \overline{\{p \in M : \omega(p) \neq 0\}}$ is compact.

then

$$\int_M d\omega = \int_{\partial M} \iota^* \omega \qquad \iota : \partial M \to M$$

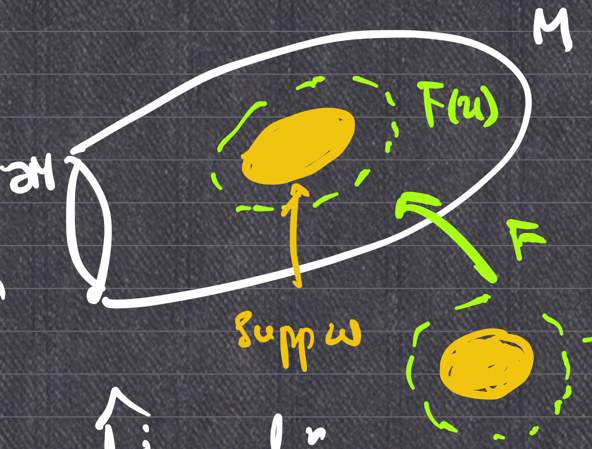
Proof:

Step ①: $\operatorname{supp}\omega \subset \underbrace{F(U)}$ ← local coordinate chart of interior type.

Step ②: $\operatorname{supp}\omega \subset G(V)$ ← local coordinate chart of boundary type

Step ③: Use partition of unity to prove the general case.

Step ①: Suppose $\operatorname{supp}(\omega) \subset F(\mathcal{U})$, $F(u_1, \dots, u_n)$



$$\int_M d\omega = \int_{F(\mathcal{U})} d\omega$$

let $\omega = \sum_{j=1}^{n} \omega_j \, du^1 \wedge \dots \wedge \widehat{du^j} \wedge \dots du^n$

$$d\omega = \sum_{j=1}^{n} \underbrace{\sum_{i=1}^{n} \frac{\partial \omega_j}{\partial u_i} du^i}_{d\omega_j} \wedge \underbrace{du^1 \wedge \dots \wedge \widehat{du^j} \wedge \dots \wedge du^n}_{\neq 0 \text{ only when } i=j}$$

$$= \sum_{i=1}^{n} \frac{\partial \omega_i}{\partial u_i} du^i \wedge du^1 \wedge \dots \wedge \widehat{du^i} \wedge \dots \wedge du^n$$

$$= \left(\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} \right) du^1 \wedge \dots \wedge du^n$$

$$\int_M d\omega = \pm \int_{\mathcal{U}} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \dots du^n$$

$$= \pm \int_{[-R,R]^n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \dots du^n$$

$$= \pm \sum_{i=1}^{n} \int_{-R}^{R} \int_{-R}^{R} \dots \int_{-R}^{R} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^1 \dots du^n.$$

$$= \pm \sum_{i=1}^{n} \int_{-R}^{R} \int_{-R}^{R} \dots \boxed{\int_{-R}^{R} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i} du^i} du^1 \dots \widehat{du^i} \dots du^n.$$

$$= (-1)^{i-1} \left[\omega_i(u_1, \dots, u_n) \right]_{u_i=-R}^{u_i=R} = \overset{0}{(-1)^{i-1} \cancel{\omega_i(u_1, \dots, u_{n-1}, R)}} - (-1)^{i-1} \cancel{\omega_i(u_1, \dots, u_{n-1}, -R)} \; 0$$

$$= 0.$$

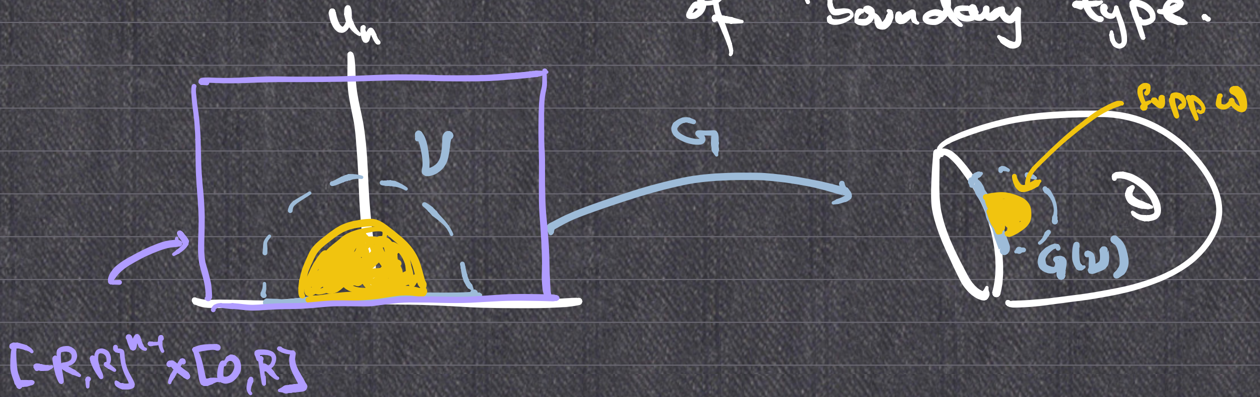
$\omega = 0$ on ∂M.

$$\int_{\partial M} \omega = 0$$

$$\therefore \int_M d\omega = \int_{\partial M} \omega = 0 \quad (\text{in this case})$$

$\operatorname{supp}\omega \subset F(\mathcal{U})$ ↑ interior.

Step ② : $\operatorname{supp}\omega \subset G(\mathcal{V})$ ↑ local parametrization of boundary type.



$$\int_M d\omega = \int_{[-R,R]^{n-1}\times[0,R]} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial u_i}\, du^1 \dots du^n$$

$$= \left(\sum_{i=1}^{n-1} (-1)^{i-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_i}{\partial u_i}\, du^1 \dots du^n \right) = 0 \text{ by similar argument as in Step ①.}$$

$$+ (-1)^{n-1} \int_0^R \int_{-R}^R \cdots \int_{-R}^R \frac{\partial \omega_n}{\partial u_n}\, du^1 \dots du^n.$$

⋮

Read it!

Step 3: $\{F_\alpha : \mathcal{U}_\alpha \to M\}$ oriented atlas of M.

$\rightsquigarrow \{\rho_\alpha : M \to [0,1]\}$ partition by unity

$\operatorname{supp} \rho_\alpha \subset F_\alpha(\mathcal{U}_\alpha)$.

$$\int_M d\omega := \sum_\alpha \int_M \underbrace{\rho_\alpha d\omega}$$

$\operatorname{supp}(\rho_\alpha d\omega) \subset F_\alpha(\mathcal{U}_\alpha)$

$$= \sum_\alpha \int_{F_\alpha(\mathcal{U}_\alpha)} \rho_\alpha d\omega = \sum_\alpha \int_{F_\alpha(\mathcal{U}_\alpha)} d(\rho_\alpha \omega) - d\rho_\alpha \wedge \omega.$$

$$= \sum_\alpha \int_M d(\rho_\alpha \omega) - d\rho_\alpha \wedge \omega$$

$$= \sum_\alpha \int_{\partial M} \iota^*(\rho_\alpha \omega) \quad - \int_M \sum_\alpha d\rho_\alpha \wedge \omega$$

$$= \sum_\alpha \int_{\partial M} (\rho_\alpha \circ \iota)\, \iota^* \omega \quad - \cancel{\int_M d(\underbrace{\textstyle\sum_\alpha \rho_\alpha}_{1}) \wedge \omega}$$

$$= \int_{\partial M} \iota^* \omega$$

$\blacksquare$

Recall:

$$\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k} \longleftrightarrow \alpha = P\,dx + Q\,dy + R\,dz$$

$$\nabla \times \vec{F} \longleftrightarrow d\alpha$$

$$G \longleftrightarrow \beta$$

$$G \cdot \hat{n}\, d\sigma \longleftrightarrow \iota^* \beta$$

β: 2-form

$\Sigma^2 \subset \mathbb{R}^3$

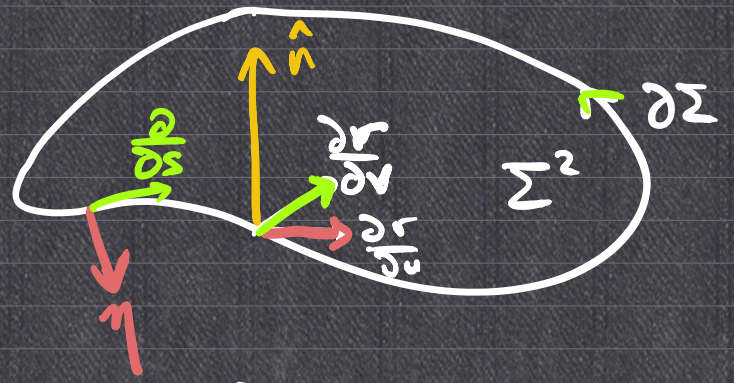
$\iota : \Sigma^2 \to \mathbb{R}^3$

$$\int_{\partial\Sigma^2} P\,dx + Q\,dy + R\,dz = \int_{\partial\Sigma^2} i^*\alpha = \int_{\Sigma^2} i^* \underbrace{d\alpha}_{\nabla\times F} = \int_{\Sigma^2} (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma$$

$$(i_\eta\Omega)\left(\frac{\partial}{\partial s}\right) > 0 .$$

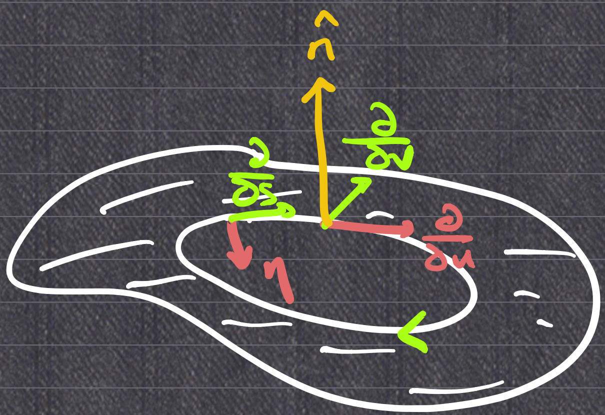
$$\Omega\left(\eta, \frac{\partial}{\partial s}\right) > 0$$

$$= \int_{\Sigma^2} (\nabla\times\vec{F})\cdot \underbrace{\frac{\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}}{\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right|}}_{\hat{n}} \cdot \underbrace{\left|\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right| du\wedge dv}_{d\sigma}$$



If we use $\Omega = du\wedge dv$ as an orientation on Σ.

$\Omega\left(\eta, \frac{\partial}{\partial s}\right)$, $\Omega\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$ both $\oplus$



$$\Omega = du\wedge dv$$

$$(i_\eta\Omega)\left(\frac{\partial}{\partial s}\right) > 0$$

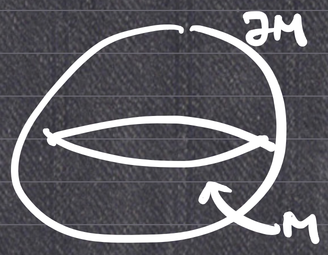
$$\Leftrightarrow \Omega\left(\underline{\eta}, \frac{\partial}{\partial s}\right) > 0$$

β two-form in $\mathbb{R}^3$ $\longleftrightarrow$ $\vec{G}$ in $\mathbb{R}^3$

$d\beta$ $\longleftrightarrow$ $\nabla\cdot\vec{G}$

$$\int_{\partial M}\beta = \int_M d\beta$$

$$\int_{\partial M}\vec{G}\cdot\hat{n}\,d\sigma = \int_M \nabla\cdot\vec{G}\,dx\wedge dy\wedge dz .$$



Choose orientation of $\mathbb{R}^3$ (hence also on M):

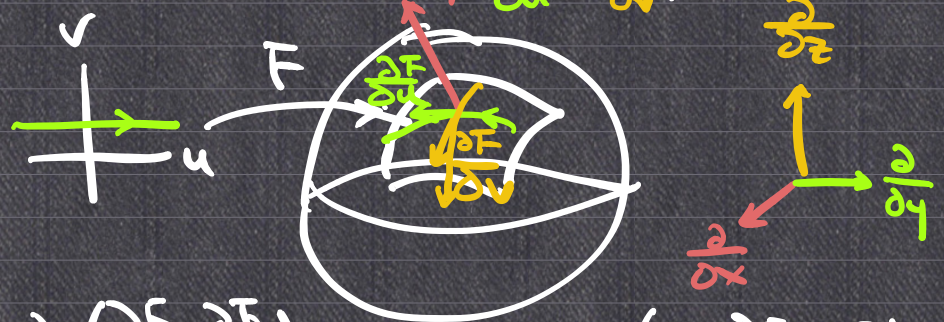
$$\Omega = dx \wedge dy \wedge dz.$$

$$\int_M \nabla\cdot\vec{G}\, dx\wedge dy\wedge dz = \int_M \nabla\cdot\vec{G}\, dx\,dy\,dz$$

$$\Omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) > 0.$$

$\rightarrow$ $i_\eta \Omega$ is the orientation of ∂M.

$\eta \,/\!/\, \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \,/\!/\, \hat{n}$



need: $\left(i_\eta \Omega\right)\left(\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right) > 0 \Leftrightarrow \Omega\left(\eta, \frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right) > 0.$

then $$\int_{\partial M} G\cdot\hat{n}\, d\sigma = \int_{\partial M} \vec{G}\cdot \underbrace{\frac{\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}}{\left|\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right|}}_{\hat{n}} \underbrace{\left|\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right| du\wedge dv}_{d\sigma}$$