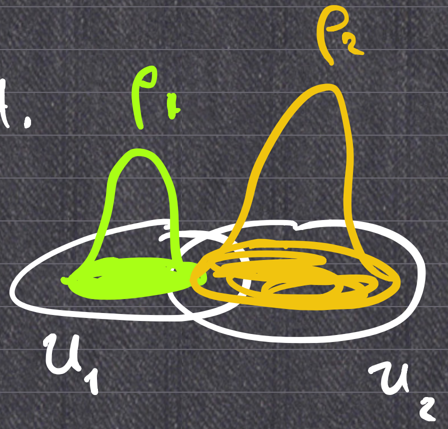
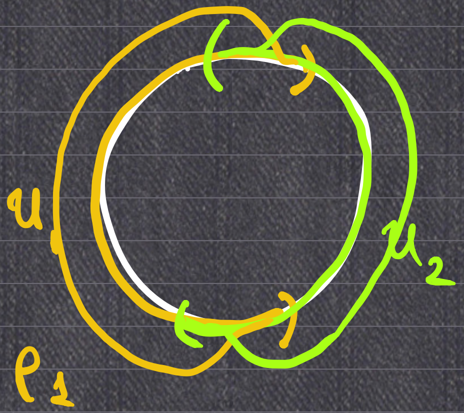
M compact manifold

$\mathcal{A} = \{ F_i : U_i \to M \}_{i=1}^{N}$

then $\exists \, \{ \rho_i : M \to [0,1] \}$ C^∞ s.t.

- $\operatorname{supp} \rho_i \subset U_i$
- $\sum_{i=1}^{N} \rho_i \equiv 1$ on M.





$$\underset{\substack{\uparrow \\ \text{n-form}}}{\int_{M^n} \omega} := \underbrace{\sum_\alpha \int_{F_\alpha(U_\alpha)} \rho_\alpha \omega}$$

$\rho_\alpha \equiv 0$ outside $F_\alpha(U_\alpha)$

1. indep. of oriented atlas ?
2. indep. of partition of unity.

$\{ F_\alpha : U_\alpha \to M \}$ finite oriented altas

$\rightsquigarrow \{ \rho_\alpha : M \to [0,1] \}$

$\{ G_\beta : V_\beta \to M \}$ finite oriented altas

$\longrightarrow \{ \sigma_\beta : M \to [0,1] \}$

$$\sum_\alpha \int_{F_\alpha(U_\alpha)} \rho_\alpha \omega = \sum_\alpha \int_{F_\alpha(U_\alpha)} \underbrace{\left(\sum_\beta \sigma_\beta\right)}_{\equiv 1} \rho_\alpha \omega$$

$$= \sum_\alpha \sum_\beta \int_{F_\alpha(U_\alpha)} \sigma_\beta \rho_\alpha \omega$$

$$= \sum_\alpha \sum_\beta \int_{F_\alpha(U_\alpha) \cap G_\beta(V_\beta)} \sigma_\beta \rho_\alpha \omega$$

$\operatorname{supp} \sigma_\beta \subset G_\beta(V_\beta)$

$$\sum_\beta \int_{G_\beta(V_\beta)} \sigma_\beta \omega = \cdots\cdots = \sum_\alpha \sum_\beta \int_{F_\alpha(U_\alpha) \cap G_\beta(V_\beta)} \sigma_\beta \rho_\alpha \omega.$$

$\{F_\alpha(u_\alpha^1, \ldots, u_\alpha^n) : U_\alpha \to M\}$ oriented atlas.

$$\int_M \sum_i \omega_i \, du_\alpha^1 \wedge \cdots \wedge du_\alpha^n := \sum_\alpha \int_{U_\alpha} \sum_i \rho_\alpha \omega_i \, du_\alpha^1 \cdots du_\alpha^n$$

<u>Proposition.</u>

A manifold M^n is orientable

$\Leftrightarrow$ $\exists$ a smooth non-vanishing n-form Ω on M.
(global) ($\Omega(p) \neq 0 \quad \forall p \in M$)

<u>Proof:</u>

($\Longrightarrow$) $\{F_\alpha : U_\alpha \to M\}$ oriented atlas of M.

$F_\alpha(u_\alpha^1, \dots, u_\alpha^n)$

$du_\alpha^1 \wedge \dots \wedge du_\alpha^n$ non-vanishing on $F_\alpha(U_\alpha)$.

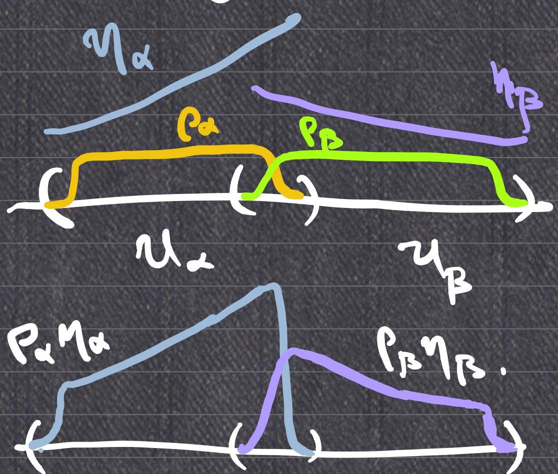
$$\cancel{\Omega := du_\alpha^1 \wedge \dots \wedge du_\alpha^n \text{ on } F_\alpha(U_\alpha) \quad \forall \alpha.} \qquad ?$$

not indep. of coordinates.

$$\Omega := \sum_\alpha \overbrace{\underbrace{\rho_\alpha}_{\substack{=0 \\ \text{outside} \\ F_\alpha(U_\alpha)}} \underbrace{du_\alpha^1 \wedge \dots \wedge du_\alpha^n}_{\eta_\alpha \text{, only make sense on } F_\alpha(U_\alpha)}}^{\text{make sense on } M} \quad \text{on } M.$$

<u>Claim:</u> Ω is C^∞ non-vanishing n-form on M.

<u>Proof:</u> Fix β. $F_\beta : U_\beta \to M$



$$\sum_\alpha \rho_\alpha \, du_\alpha^1 \wedge \dots \wedge du_\alpha^n$$

$$= \sum_\alpha \rho_\alpha \det \frac{\partial(u_\alpha^1, \dots, u_\alpha^n)}{\partial(u_\beta^1, \dots, u_\beta^n)} du_\beta^1 \wedge \dots \wedge du_\beta^n$$

$$= \Bigg(\sum_\alpha \underbrace{\rho_\alpha}_{\geq 0} \underbrace{\det \frac{\partial(u_\alpha^1, \dots, u_\alpha^n)}{(u_\beta^1, \dots, u_\beta^n)}}_{>0 \text{ (oriented atlas)}} \Bigg) du_\beta^1 \wedge \dots \wedge du_\beta^n \qquad \neq 0.$$

$\sum \rho_\alpha \equiv 1 \Rightarrow$ (not all $\rho_\alpha = 0$)

($\Leftarrow$) Given Ω.

$\{G_\alpha : V_\alpha \to M\}$ any atlas, $G_\alpha(v_\alpha^1, \dots, v_\alpha^n)$

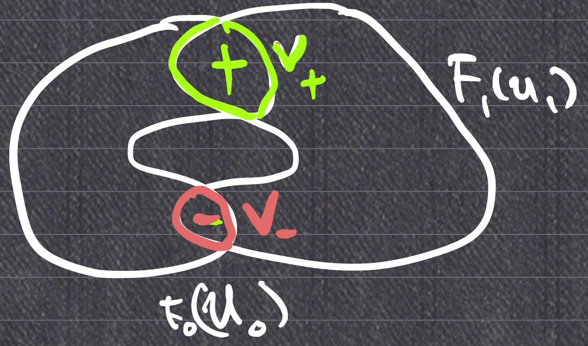
$$\rightsquigarrow F_\alpha(u_\alpha^1, \dots, u_\alpha^n) := \begin{cases} G_\alpha(u_\alpha^1, \dots, u_\alpha^n) & \text{if } \Omega\left(\frac{\partial}{\partial v_\alpha^1}, \dots, \frac{\partial}{\partial v_\alpha^n}\right) > 0 \\ G_\alpha(-u_\alpha^1, \dots, u_\alpha^n) & \text{if } \Omega\left(\frac{\partial}{\partial v_\alpha^1}, \dots, \frac{\partial}{\partial v_\alpha^n}\right) < 0 \end{cases}$$

$\Rightarrow$ $\{F_\alpha : U_\alpha \to M\}$ is oriented atlas.

<u>Cor</u>: $\mathbb{RP}^2$ is not orientable.

<u>Proof</u>: Done before:

$$\det D(F_0^{-1} \circ F_1) = \begin{cases} + & \text{on } V_+ \\ - & \text{on } V_- \end{cases}$$



$F_0(u_1, u_2)$

$F_1(v_1, v_2)$

Assume otherwise $\mathbb{RP}^2$ is orientable

$\Rightarrow$ $\exists$ C^∞ non-vanishing 2-form Ω.

① $\Omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) > 0$ on $F_0(U_0)$

$$\Rightarrow \Omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) = \det \frac{\partial(u_1, u_2)}{\partial(v_1, v_2)} \, \Omega\left(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}\right)$$

V_+	+	+	+
V_-	+	$-$	$-$

$\Omega\left(\frac{\partial}{\partial v_1}, \frac{\partial}{\partial v_2}\right)$ $\quad \begin{matrix} - \\ + \end{matrix} \quad \} = 0.$

$= 0 \quad \exists\, p \in F_1(U_1)$

② $\Omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) < 0$ on $F_0(U_0)$

Similar.

M^n orientable manifold

$\mathcal{A}$ oriented altas $\{\det D(F_\alpha^{-1} \circ F_\beta) > 0 \quad \forall \alpha, \beta\}$.

Ω orientation.

$\uparrow$

C^∞, non-vanishing n-form

$\{T_1, \dots, T_n\}$. $\quad \Omega(T_1, \dots, T_n)$

$\{V_1, \dots, V_n)$ $\quad \underbrace{\Omega(V_1, \dots, V_n)}$ different sign

$\Downarrow$

$\{T_1, \dots, T_n\}$, $\{V_1, \dots, V_n\}$

different orientation.

$\mathbb{R}^3$

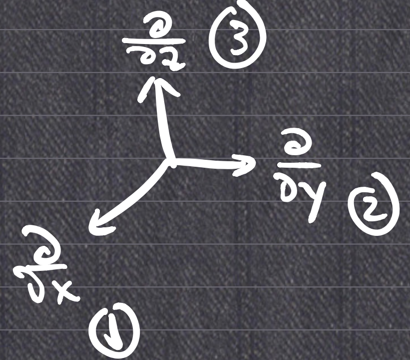
$\Omega = dx \wedge dy \wedge dz$ $\quad$ v, w linearly indep.

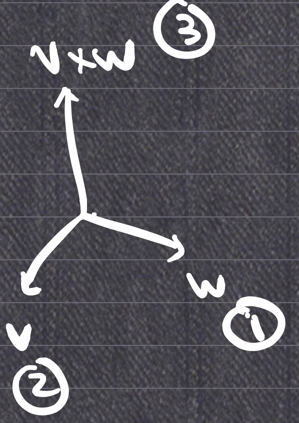
$\Omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = 1 > 0.$ $\quad$ $\Omega(v, w, v \times w) = \begin{vmatrix} | & | & | \\ v & w & v\times w \\ | & | & | \end{vmatrix}$

$= (v\times w)\cdot(v\times w) = |v\times w|^2$.

> 0.

$\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\}$ same orientation as $\{v, w, v\times w\}$.





$\{w, v, v\times w\}$

Choose Ω an orientation of M.

$$\int_{F_\alpha(U_\alpha)} f\, du^1\wedge\cdots\wedge du^n = \begin{cases} \int_{U_\alpha} f\, du^1\cdots du^n & \text{if } \Omega\left(\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_n}\right) > 0 \\ -\int_{U_\alpha} f\, du^1\cdots du^n & \text{if } \Omega\left(\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_n}\right) < 0. \end{cases}$$

M orientable manifold with boundary.

Fix Ω as an orientation of M. $\int_M d\omega = \int_{\partial M}\omega$.

$\rightsquigarrow$ $i_\eta\Omega$ as an orientation of ∂M.

η: outward normal

<u>Suppose</u> $\Omega\left(\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_n}\right) > 0$.

$$(i_\eta \Omega)\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n-1}}\right)$$

$$= \Omega\left(-\frac{\partial}{\partial u_n}, \frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n-1}}\right) \qquad \eta = -\frac{\partial}{\partial u_n}.$$

$$= -1 \cdot (-1)^{n-1}\, \Omega\left(\frac{\partial}{\partial u_1}, \dots, \frac{\partial}{\partial u_{n-1}}, \frac{\partial}{\partial u_n}\right)$$

$$= (-1)^n \, +$$

$$\int_{\partial M} f\, du^1 \wedge \dots \wedge du^{n-1} = (-1)^n \int_{\partial M} f\, du^1 \cdots du^{n-1}.$$