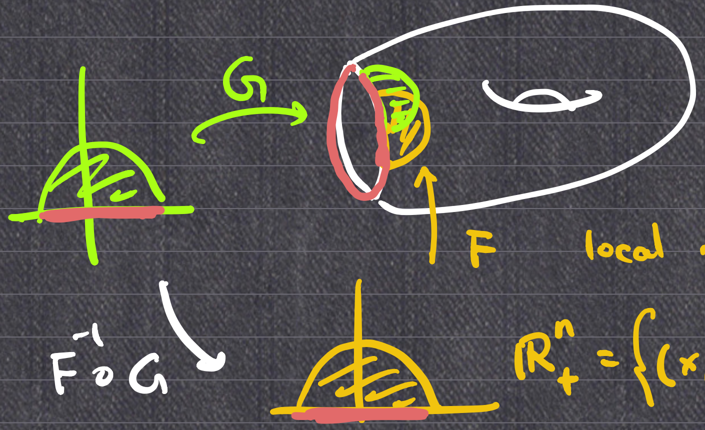


local parametrization of boundary type.

$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) : x_n \geq 0\}$.

<u>Definition</u> M is called a C^∞ manifold with boundary

$\Leftrightarrow \exists$ two families of local parametrizations

$\{F_\alpha : \mathcal{U}_\alpha \subset \mathbb{R}^n \to M\}$ interior type

$\{G_\beta : \mathcal{V}_\beta \subset \mathbb{R}^n_+ \to M\}$. boundary type

s.t. $\left(\bigcup_\alpha F_\alpha(\mathcal{U}_\alpha)\right) \cup \left(\bigcup_\beta G_\beta(\mathcal{V}_\beta)\right) = M$

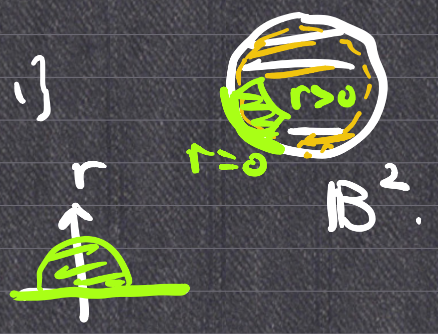
and $F_\alpha^{-1} \circ F_\beta$, $F_\alpha^{-1} \circ G_\beta$, $G_\beta^{-1} \circ G_\alpha$, $G_\beta^{-1} \circ F_\alpha$

are all C^∞.

$$\partial M := \bigcup_\beta \left\{ G_\beta(v_1, \dots, v_{n-1}, 0) : (v_1, \dots, v_{n-1}, 0) \in \mathcal{V}_\beta \right\}.$$

<u>e.g.</u> $\mathbb{B}^2 = \{\vec{x} \in \mathbb{R}^2 : |\vec{x}| \leq 1\}$

interior: $\mathrm{id} : \{|x| < 1\} \to \{|x| < 1\}$



polar: $x = (-r+1)\cos\theta$
$y = (-r+1)\sin\theta$

boundary type parametrizations:

$$G_1(\theta, r) = \big((1-r)\cos\theta,\ (1-r)\sin\theta\big) : (0,2\pi)\times[0,\tfrac{1}{2}) \to B^2$$

$$G_2(\theta, r) = \big((1-r)\cos\theta,\ (1-r)\sin\theta\big) : (-\pi,\pi)\times[0,\tfrac{1}{2}) \to B^2.$$

id $\quad G_1 \quad G_2$

✓ $id^{-1}\circ G_1 = G_1 \rightsquigarrow \widetilde{G_1}(\theta, r) = \big((1-r)\cos\theta,\ (1-r)\sin\theta\big)$

✓ $id^{-1}\circ G_2 = G_2$

$: (0,2\pi)\times(-\frac{1}{2},\frac{1}{2}) \to \mathbb{R}^2.$

is C^∞

$\Rightarrow G_1$ is C^∞ on $(0,2\pi)\times[0,\frac{1}{2})$.

✓ $G_1^{-1}\circ id$

✓ $G_2^{-1}\circ id$

> inverse function theorem.

$G_2^{-1}\circ G_1(\theta,r) : \underbrace{G_1^{-1}\Big(}$  $\Big) \to \mathbb{R}^2$

$= \big((0,\pi)\times[0,\tfrac{1}{2})\big) \amalg \big((\pi,2\pi)\times[0,\tfrac{1}{2})\big)$

$$G_2^{-1}\circ G_1(\theta,r) = \begin{cases} (\theta, r) & \text{if } \theta\in(0,\pi) \\ (\theta-2\pi, r) & \text{if } \theta\in(\pi,2\pi) \end{cases}$$

extension: $\widetilde{G_2^{-1}\circ G_1}(\theta,r) := \begin{cases} (\theta, r) & \text{if } \theta\in(0,\pi) \\ (\theta-2\pi, r) & \text{if } \theta\in(\pi,2\pi) \end{cases}$ is C^∞.

$\theta\in(0,\pi)\amalg(\pi,2\pi) \qquad (-\frac{1}{2},\frac{1}{2})$

$\Rightarrow G_2^{-1}\circ G_1$ is C^∞ ✓

<u>Exercise</u>: $G_1^{-1}\circ G_2$.

Prop:

Let $f: M^m \to \mathbb{R}$ be C^∞ function.

Suppose $\Sigma = \underbrace{f^{-1}([c,\infty))}_{} \neq \emptyset$. $\quad \underbrace{f^{-1}((c,+\infty))}_{\text{open}}$.

$= \{p \in M : f(p) \geq c\}$.

If f is a submersion at $\forall p \in f^{-1}(c)$

then Σ is a m-manifold with boundary

and $\partial\Sigma = f^{-1}(c)$.

e.g. $\mathbb{B}^n := \{\vec{x} \in \mathbb{R}^n : |\vec{x}| \leq 1\}$ is a

n-manifold with boundary, and $\partial\mathbb{B}^n = \{|\vec{x}| = 1\}$.

Proof: $f(\vec{x}) := -|\vec{x}|^2$.

$$\mathbb{B}^n = f^{-1}([-1,+\infty))$$

Need: f is a submersion on $f^{-1}(-1)$.

$= \{|\vec{x}| = 1\}$.

$$\nabla f = (-2x_1, -2x_2, \dots, -2x_n) = 0$$

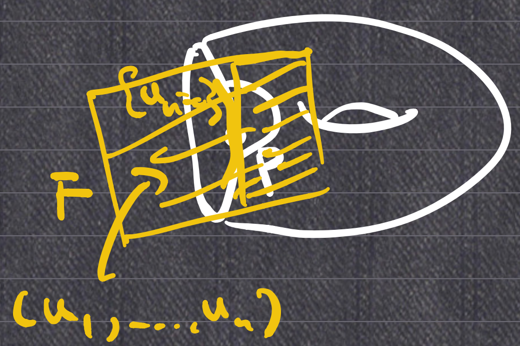
$$\iff \vec{x} = 0.$$

$\therefore \nabla f(\vec{x}) \neq 0 \quad \forall \vec{x} \in f^{-1}(-1)$.

$\Rightarrow f$ is a submersion on $f^{-1}(-1)$.

- $p \in \partial M$, $\quad T_pM := \operatorname{span}\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^{n}$

$$= \left\{\sum_{i=1}^{n} a^i \frac{\partial}{\partial u_i}(p) : a^i \in \mathbb{R}\right\}.$$

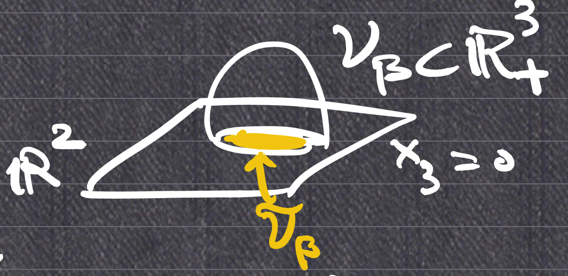


- M^n manifold with boundary,

∂M is a $(n-1)$-manifold without boundary.

$$\partial M = \bigcup_{\beta} \left\{ G_\beta(u_1,\ldots,u_{n-1},0) : (u_1,\ldots,u_{n-1},0) \in \mathcal{V}_\beta \right\}.$$

$\beta \leftarrow$ boundary type for M. $\quad \mathbb{R}^n_+$.



$$\tilde{G}_\beta(u_1,\ldots,u_{n-1}) := G_\beta(u_1,\ldots,u_{n-1},0) : \tilde{\mathcal{V}}_\beta \to \partial M$$

$\tilde{\mathcal{V}}_\beta$ open $\subset \mathbb{R}^{n-1}$

$$\tilde{\mathcal{V}}_\beta = \{(u_1,\ldots,u_{n-1}) \in \mathbb{R}^{n-1} : (u_1,\ldots,u_{n-1},0) \in \mathcal{V}_\beta\}$$

$$\partial M = \bigcup_{\beta} \tilde{G}_\beta(\tilde{\mathcal{V}}_\beta)$$

interior type for ∂M.

$$\langle M^n, \omega \rangle := \int_M \omega \qquad (\omega \text{ n-form})$$

Stokes: $\displaystyle\int_{\partial M} \omega = \int_M d\omega \qquad (\omega \text{ n-1})$

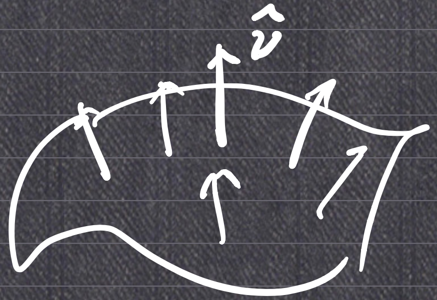
$$\langle \partial M, \omega \rangle = \langle M, d\omega \rangle.$$

$$\partial(\partial M) = \phi$$

$\forall\omega.\quad \langle \partial(\partial M), \omega\rangle = \langle \partial M, d\omega\rangle = \langle M, \underbrace{d^2\omega}_{=0}\rangle = 0.$

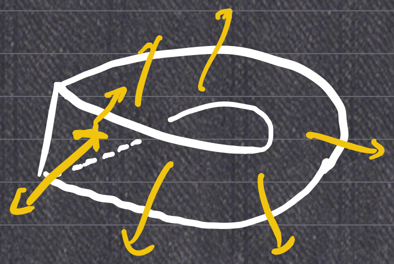
$$\text{"}\Rightarrow\text{"}\quad \partial(\partial M) = \phi.$$

§4.2 - Orientability.



$M^n \subset \mathbb{R}^{n+1}$ is orientable

$\left(\overset{def}{\iff} \exists\, \hat{\nu} : M \to \mathbb{R}^{n+1}$ is continuous.
$p \mapsto$ unit normal at p to $M^n \right)$



$\iff \exists$ family of parametrizations $\{F_\alpha : U_\alpha \to M\}$
s.t. $\bigcup_\alpha F_\alpha(U_\alpha) = M$

and $\det D(F_\alpha^{-1} \circ F_\beta) > 0 \quad \forall \alpha, \beta.$

$\iff \exists$ non-vanishing C^∞ n-form Ω.

Password: doitbyyourself