S M T W Th F S

30/3

8/4 release midterm

11/4

18/4 due date.

$$d(\omega_{i_1\cdots i_k}du^{i_1}\wedge\cdots\wedge du^{i_k}) = d\omega_{i_1\cdots i_k}\wedge du^{i_1}\wedge\cdots\wedge du^{i_k}$$

$d\omega =$ in terms of global quantities.

$$\mathcal{L}_X Y = [X,Y], \quad \mathcal{L}_X \alpha := \sum_{i,j}\left(X^i\frac{\partial \alpha_j}{\partial u_i} + \alpha_i \frac{\partial X^i}{\partial u_j}\right)du^j$$

$\alpha$: 1-form.

$S, T$ : 1-form

$$\mathcal{L}_X(S\otimes T)_p := \frac{d}{dt}\Big|_{t=0}(\Phi_t)^*(S\otimes T)_{\Phi_t(p)}$$

$$= \frac{d}{dt}\Big|_{t=0}(\Phi_t^* S)_{\Phi_t(p)}\otimes(\Phi_t^* T)_{\Phi_t(p)}$$

$$= (\mathcal{L}_X S)\otimes T + S\otimes(\mathcal{L}_X T).$$

Cartan's Magic Formula:

$\omega$ k-form, $X$ vector field.

then: $\mathcal{L}_X\omega = i_X(d\omega) + d(i_X\omega)$

where $i_X : \Lambda^k \to \Lambda^{k-1}$

$$(i_X\omega)(Y_1,\dots,Y_{k-1}) := \omega(X, Y_1,\dots,Y_{k-1})$$

k-form

(k-1) form.

Corollary: $\underbrace{(d\omega)}_{\text{k-form}}(Y_1,\dots,Y_{k+1}) = (i_{Y_1} d\omega)(Y_2,\dots,Y_{k+1})$

$= (\mathcal{L}_X\omega)(Y_2,\dots,Y_{k+1})$

$- \left(d(i_X\omega)\right)(Y_2,\dots,Y_{k+1})$

$= \dots$

Prove: $\mathcal{L}_X\omega = i_X(d\omega) + d(i_X\omega)$

let $\omega = \omega_i du^i$, $X = X^i \frac{\partial}{\partial u_i}$

$LHS = \left(\frac{\partial \omega_j}{\partial u_i} X^i + \omega_i \frac{\partial X^i}{\partial u_j}\right) du^j$

$i_X(d\omega) = i_X\left(\frac{\partial \omega_i}{\partial u_j} du^j \wedge du^i\right)$

$= \frac{\partial \omega_i}{\partial u_j}\, \underline{i_X(du^j \wedge du^i)}$

✓

$\left(i_X(du^j \wedge du^i)\right)\left(\frac{\partial}{\partial u_k}\right) = (du^j \wedge du^i)\left(X, \frac{\partial}{\partial u_k}\right)$

$= (du^j \wedge du^i)\left(X^l \frac{\partial}{\partial u_l}, \frac{\partial}{\partial u_k}\right)$

$= X^l \left(du^j \otimes du^i - du^i \otimes du^j\right)\left(\frac{\partial}{\partial u_l}, \frac{\partial}{\partial u_k}\right)$

$= X^l \left(\delta_{lj}\delta_{ik} - \delta_{il}\delta_{jk}\right)$

$= X^j \delta_{ik} - X^i \delta_{jk}$

$$\Rightarrow \quad i_X (du^j \wedge du^i) = (X^j \delta_{ik} - X^i \delta_{jk})\, du^k$$

$$i_X(d\omega) = \frac{\partial \omega_i}{\partial u_j} (X^j \delta_{ik} - X^i \delta_{jk})\, du^k$$

$$= \frac{\partial \omega_i}{\partial u_j} X^j du^i - X^i \frac{\partial \omega_i}{\partial u_j} du^j$$

$$d(\underbrace{i_X \omega}) = d(\underbrace{\omega(X)}_{\text{scalar}})$$

$(i_X \omega) = \omega(X)$

1-form.

$$= d\left((\omega_i du^i)(X^j \tfrac{\partial}{\partial u_j})\right)$$

$$= d(\omega_i X^j \delta_{ij}) = d(\omega_i X^i)$$

$$= \frac{\partial(\omega_i X^i)}{\partial u_j} du^j$$

$$= \left(\frac{\partial \omega_i}{\partial u_j} X^i + \frac{\partial X^i}{\partial u_j} \omega_i\right) du^j$$

$$i_X d\omega + d i_X \omega = \frac{\partial \omega_i}{\partial u_j} X^j du^i \cancel{- X^i \frac{\partial \omega_i}{\partial u_j} du^j}$$

$$\left(\cancel{\frac{\partial \omega_i}{\partial u_j} X^i} + \frac{\partial X^i}{\partial u_j} \omega_i\right) du^j$$

$$= \frac{\partial \omega_j}{\partial u_i} X^i du^j + \frac{\partial X^i}{\partial u_j} \omega_i \, du^j$$

$= \text{LHS}.$

$$\underset{\sim}{\mathcal{L}_X du^i} = d(i_X du^i) + i_X\cancel{(d(du^i))}$$

$$= d(du^i(X))$$

$$= d\left(du^i\left(X^j \frac{\partial}{\partial u_j}\right)\right)$$

$$= d(X^i)$$

$$= \frac{\partial X^i}{\partial u_j} du^j$$

$$\mathcal{L}_{\frac{\partial}{\partial x}}(y^2 dx) \qquad \text{in } \mathbb{R}^2$$

$$= \frac{\partial y^2}{\partial x} dx + y^2 \underbrace{\mathcal{L}_{\frac{\partial}{\partial x}} dx}_{} = 0.$$

$$= \cancel{i_{\frac{\partial}{\partial x}} d(dx)} + d\, i_{\frac{\partial}{\partial x}} dx$$

$$= d(1) = 0.$$

$$\mathcal{L}_{\frac{\partial}{\partial y}}(y^2 dx \otimes dy)$$

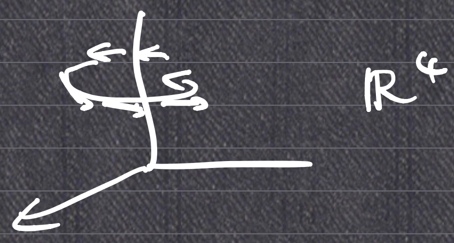
$$= \frac{\partial(y^2)}{\partial y} dx \otimes dy + y^2 \underbrace{\left(\mathcal{L}_{\frac{\partial}{\partial x}} dx\right)}_{} \otimes dy + y^2 dx \otimes \underbrace{\mathcal{L}_{\frac{\partial}{\partial y}} dy}_{}.$$

use Cartan's formula

$$g = -\left(1-\frac{r_s}{r}\right)c^2dt^2 + \frac{1}{1-\frac{r_s}{r}}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

(Schwarzschild's space-time)

$\frac{\partial}{\partial\theta}$



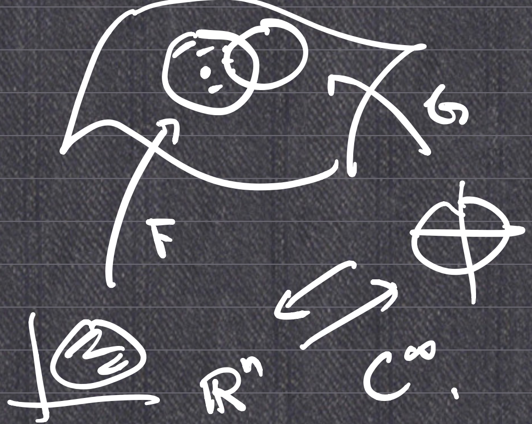

$$\boxed{\mathcal{L}_{\frac{\partial}{\partial\theta}} g = 0} \overset{\text{def}}{\iff}$$ space-time $(M, g)$ is invariant along the direction of $\frac{\partial}{\partial\theta}$.

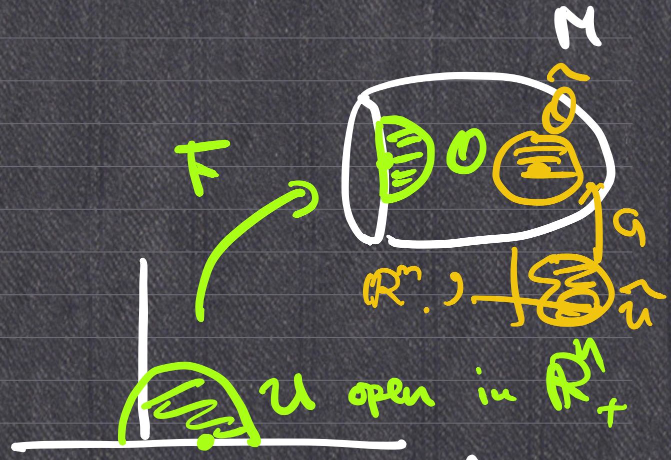
$$\Updownarrow$$

$$\boxed{\Phi_t^* g = g.}$$

§4

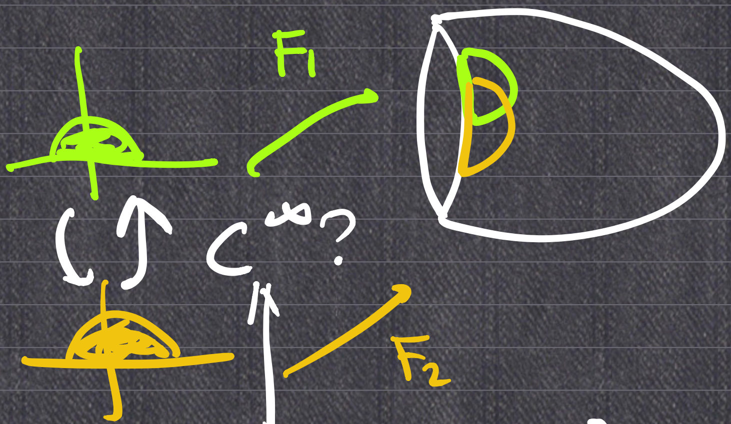
Manifold with boundary.




Manifold without boundary.

$\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) : x_n \geq 0\}$.




what it mean?

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$\phi : \mathbb{R}^n_+ \to \mathbb{R}^m$ is $C^k$

$\overset{\text{def}}{\iff}$ $\phi$ is $C^k$ on $\mathbb{R}^n_+ \setminus \{x_n\text{-axis}\}$

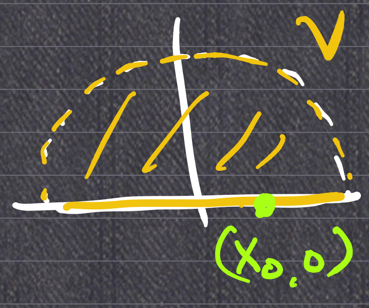
and $\forall p \in \partial \mathbb{R}^n_+ = \{(x_1, \dots, x_{n-1}, 0)\}$

$\exists B_\varepsilon(p) \subset \mathbb{R}^n$

and $\tilde{\Phi} : B_\varepsilon(p) \to \mathbb{R}^m$ s.t.


$\tilde{\Phi}$ is $C^k$ on $B_\varepsilon(p)$ and $\tilde{\Phi} = \phi$ on $\mathbb{R}^n_+ \cap \partial B_\varepsilon(p)$.

e.g. $V = \{(x,y) : x^2+y^2 < 1,\ y \geq 0\}$




$f(x,y) : V \to \mathbb{R}$

$f(x,y) = \sqrt{1-x^2-y^2}$. is $C^\infty$ on $V$.

- $f$ is $C^\infty$ on $V^\circ = \{(x,y) : x^2+y^2 < 1,\ y > 0\}$.
- $\forall (x_0, 0) \in \partial V$, define $\tilde{f}(x,y) = \sqrt{1-x^2-y^2}$.

$\tilde{f} : \underset{\underset{\substack{B_\varepsilon(p) \\ \nwarrow \text{small}}}{\cup}}{B_0(1)} \to \mathbb{R}$ is $C^\infty$.

$\therefore \tilde{f} = f$ on $B_0(1) \cap \mathbb{R}^n_+$.

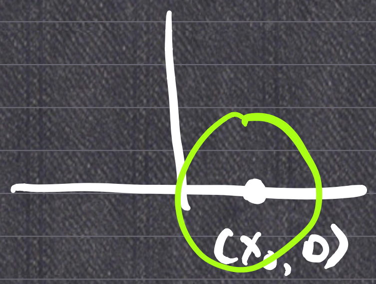
$\Rightarrow f$ is $C^\infty$ on $\partial V$.

$\therefore f$ is $C^\infty$ on $V$.

<u>e.g.</u> $g(x,y) = |y| : \mathbb{R}^2_+ \to \mathbb{R}$

<u>Claim</u>: $g$ is $C^\infty$ on $\partial \mathbb{R}^2_+$.

<u>Proof</u>




Define $\tilde{g}(x,y) = y$.

$\tilde{g}(x,y) : \mathbb{R}^2 \to \mathbb{R}$ is $C^\infty$.

$\underbrace{\tilde{g}(x,y)}_{y} = \underbrace{g(x,y)}_{|y|}$ on $\underbrace{\mathbb{R}^n_+}_{y \geq 0}$.

$\therefore g(x,y) = |y|$ is $C^\infty$ on $\partial \mathbb{R}^2_+$.