Multivariable	4033
f	f
∇f	df
$\vec{F}$	1-form α
$\nabla\times\vec{F}$	$d\alpha$
$\vec{G}$	2-form β
$\nabla\cdot\vec{G}$	$d\beta$

$dx\wedge dy \leftrightarrow \hat{k}$.

$\Phi : M \to N$

$T = \sum T_{i_1\cdots i_k}\, dv^{i_1}\otimes\cdots\otimes dv^{i_k}$ on N.

$$\Phi^* T = \sum \underbrace{\Phi^* T_{i_1\cdots i_k}}_{T_{i_1\cdots i_k}\circ\Phi}\, \Phi^*(dv^{i_1})\otimes\cdots\otimes\Phi^*(dv^{i_k})$$

$$\alpha = \sum \alpha_{i_1\cdots i_k}\, dv^{i_1}\wedge\cdots\wedge dv^{i_k}$$

$$= \sum_{i_1\cdots i_k}\sum_{\sigma\in S_k} \alpha_{i_1\cdots i_k}\,\mathrm{sgn}(\sigma)\, dv^{i_{\sigma(1)}}\otimes\cdots\otimes dv^{i_{\sigma(k)}}.$$

$$\Phi^*\alpha = \sum_{i_1\cdots i_k} \Phi^*\alpha_{i_1\cdots i_k}\,\Phi^* dv^{i_1}\wedge\cdots\wedge\Phi^* dv^{i_k}$$

e.g. $\underset{(x,y)}{\Phi : \mathbb{R}^2} \to \underset{(u,v)}{\mathbb{R}^2}$, $\quad \Phi(x,y) = (\underbrace{x^2-y}_{u}, \underbrace{y^3}_{v})$.

$\Phi^*(du\wedge dv) = ?$

$$\Phi^* du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 2x\,dx - dy$$

$$\Phi^* dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 3y^2\,dy$$

$$\Phi^* du \wedge \Phi^* dv = (2x\,dx - dy) \wedge 3y^2 dy = 6xy^2\, dx \wedge dy.$$

e.g.

$$\begin{array}{ccc} \Phi : & M^n & \to N^n \\ & \nearrow & \nwarrow \\ & F(u_1, \dots, u_n) & G(v_1, \dots, v_n) \end{array}$$

$$\Phi^*(dv^1 \wedge \dots \wedge dv^n) := (\Phi^* dv^1) \wedge \dots \wedge (\Phi^* dv^n)$$

$$= \left(\sum_{i_1} \frac{\partial v^1}{\partial u_{i_1}} du^{i_1} \right) \wedge \dots \wedge \left(\sum_{i_n} \frac{\partial v^n}{\partial u_{i_n}} du^{i_n} \right)$$

$$\Phi^* dv^j = \sum_i \frac{\partial v^j}{\partial u_i} du^i$$

$$= \sum_{\substack{i_1, \dots, i_n \\ \text{distinct}}} \frac{\partial v^1}{\partial u_{i_1}} \frac{\partial v^2}{\partial u_{i_2}} \cdots \frac{\partial v^n}{\partial u_{i_n}} du^{i_1} \wedge \dots \wedge du^{i_n}$$

$$= \sum_{\sigma \in S_n} \frac{\partial v^1}{\partial u_{\sigma(1)}} \frac{\partial v^2}{\partial u_{\sigma(2)}} \cdots \frac{\partial v^n}{\partial u_{\sigma(n)}} \underbrace{du^{\sigma(1)} \wedge \dots \wedge du^{\sigma(n)}}_{\operatorname{sgn}(\sigma)\, du^1 \wedge \dots \wedge du^n}$$

$$= \det \left[\frac{\partial v^j}{\partial u_i} \right] du^1 \wedge \dots \wedge du^n$$

$$= \det \frac{\partial (v^1, \dots, v^n)}{\partial (u_1, \dots, u_n)} du^1 \wedge \dots \wedge du^n.$$

Prop:

$\Phi^*(d\omega) = d(\Phi^* \omega)$. $\forall$ k-form ω.

$$\begin{array}{ccc} M & \longrightarrow & N \\ (u_i) & & (v_\alpha) \end{array}$$

Proof: Claim: $\Phi^*(df) = d(\Phi^* f)$.

$$\text{LHS} = \Phi^*\left(\sum_\alpha \frac{\partial f}{\partial v_\alpha} dv^\alpha \right) = \sum_\alpha \left(\frac{\partial f}{\partial v_\alpha} \circ \Phi \right) \sum_j \frac{\partial v^\alpha}{\partial u_j} du^j$$

$$= \sum_{\alpha, j} \left(\frac{\partial f}{\partial v_\alpha} \circ \Phi \right) \frac{\partial v^\alpha}{\partial u_j} du^j$$

$$\text{RHS} = d(f\circ\Phi) \qquad\qquad f\circ\Phi : M \to \mathbb{R}$$

$$= \sum_i \frac{\partial (f\circ\Phi)}{\partial u_i} du^i$$

$$= \sum_i \sum_\alpha \underbrace{\frac{\partial f}{\partial v_\alpha}}_{\text{at } \Phi(u_i)} \frac{\partial v^\alpha}{\partial u_i} du^i$$

$$= \text{LHS}.$$

$f\circ\Phi$
|
N
|
M.

For k-form $\omega = f\, dv^{i_1}\wedge\cdots\wedge dv^{i_k}$

$$d\omega = df \wedge dv^{i_1}\wedge\cdots\wedge dv^{i_k}$$

$$\Phi^*(d\omega) = \Phi^*(df)\wedge \Phi^*(dv^{i_1})\wedge\cdots\wedge\Phi^*(dv^{i_k})$$

$$(\Phi^*\omega) = \Phi^*f \;\; \Phi^* dv^{i_1}\wedge\cdots\wedge\Phi^* dv^{i_k}$$

$$d(\Phi^*\omega) = d(\Phi^* f)\wedge\Phi^* dv^{i_1}\wedge\cdots\wedge\Phi^* dv^{i_k}$$
$$+ (\Phi^* f)\, d(\Phi^* dv^{i_1}\wedge\cdots\wedge\Phi^* dv^{i_k})$$
$$= \Phi^*(df)\wedge\Phi^* dv^{i_1}\wedge\cdots\wedge\Phi^* dv^{i_k}$$
$$+ (\Phi^* f)\, \cancel{d\big(d(\Phi^* v^{i_1})\wedge\cdots\wedge d(\Phi^* v^{i_k})\big)}$$

$= 0$ since $d^2 = 0$.

∎

<u>Cor:</u> Φ^*(closed) is closed

Φ^*(exact) is exact.

$$\vec{F} = \alpha_x \hat{i} + \alpha_y \hat{j} + \alpha_z \hat{k} \longleftrightarrow \alpha = \alpha_x dx + \alpha_y dy + \alpha_z dz.$$

$$\vec{r}(t) := (x(t), y(t), z(t))$$

$\gamma \subset \mathbb{R}^3$

$$\begin{tikzcd} & \gamma \arrow[r, "\iota"] & \mathbb{R}^3 \\ \mathbb{R} \arrow[ur, "\vec{r}"] \arrow[rr, "id^{-1}\circ \iota \circ \vec{r}"'] & & \mathbb{R}^3 \arrow[u, "id"'] \end{tikzcd}$$

$= (x(t), y(t), z(t))$

$$\begin{aligned} &\iota^*\alpha \\ &= \iota^*(\alpha_x dx + \alpha_y dy + \alpha_z dz) \\ &= \alpha_x (x'(t)\,dt) \\ &\quad + \alpha_y (y'(t)\,dt) + \alpha_z (z'(t)\,dt) \\ &= (\alpha_x \hat{i} + \alpha_y \hat{j} + \alpha_z \hat{k}) \cdot (x'(t), y'(t), z'(t))\,dt \\ &= \vec{F} \cdot \vec{r}\,'(t)\,dt = \vec{F} \cdot d\vec{r} \end{aligned}$$

$\Sigma^2 \subset \mathbb{R}^3$ regular surface.

$\iota : \Sigma \to \mathbb{R}^3$

$$\begin{tikzcd} & \Sigma^2 \arrow[r, "\iota"] & \mathbb{R}^3 \\ (u,v) \arrow[ur, "{F(u,v) = (x(u,v), y \ldots, z(u,v))}"] \arrow[rr] & & (x,y,z) \arrow[u, "id"'] \end{tikzcd}$$

$$\vec{G} = \beta_x \hat{i} + \beta_y \hat{j} + \beta_z \hat{k} \longleftrightarrow \beta = \beta_x\, dy \wedge dz + \beta_y\, dz \wedge dx + \beta_z\, dx \wedge dy.$$

$$\iota^*\beta = \beta_x\, \underline{\iota^*(dy \wedge dz)} + \beta_y\, \underline{\iota^*(dz \wedge dx)} + \beta_z\, \underline{\iota^*(dx \wedge dy)}$$

$$\underline{\iota^*(dy \wedge dz)} = \iota^* dy \wedge \iota^* dz = \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right) \wedge \left(\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv\right)$$

$$= \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) du \wedge dv.$$

$$\underline{\iota^*(dz \wedge dx)} = \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) du \wedge dv.$$

$$\underline{\iota^*(dx \wedge dy)} = \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du \wedge dv.$$

$$\iota^*\beta = \left\{ \begin{aligned} &\beta_x\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right) \\ &+\beta_y\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) \\ &+\beta_z\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) \end{aligned} \right\} du\wedge dv \, ,$$

$$= \left(\beta_x \det\frac{\partial(y,z)}{\partial(u,v)} + \beta_y \det\frac{\partial(z,x)}{\partial(u,v)} + \beta_z \det\frac{\partial(x,y)}{\partial(u,v)}\right) du\wedge dv \, .$$

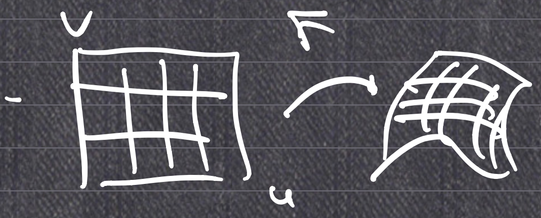
$$= \underbrace{(\beta_x\hat{i} + \beta_y\hat{j} + \beta_z)}_{\vec{G}} \cdot \underbrace{\left(\det\frac{\partial(y,z)}{\partial(u,v)}\hat{i} + \det\frac{\partial(z,x)}{\partial(u,v)}\hat{j} + \det\frac{\partial(x,y)}{\partial(u,v)}\hat{k}\right)}_{} du\wedge dv$$

$$= \begin{array}{c} \left|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{matrix}\right| \end{array} \begin{array}{l} \\ \leftarrow \frac{\partial F}{\partial u} \\ \leftarrow \frac{\partial F}{\partial v} \end{array}$$

$$= \vec{G}\cdot\left(\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right) du\wedge dv \, .$$

$$= \vec{G}\cdot \underbrace{\frac{\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}}{\left|\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right|}}_{} \underbrace{\left|\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right| du\wedge dv}_{}$$

$$= \vec{G}\cdot\hat{n} \quad d\sigma$$



Stokes' Theorem in 2023

$$\oint_C \vec{F}\cdot d\vec{r} = \int_\Sigma (\nabla\times\vec{F})\cdot\hat{n}\, d\sigma$$

$\vec{F} \leftrightarrow \underset{\omega}{\alpha}$ (1-form) $\qquad \nabla\times\vec{F} \leftrightarrow d\alpha$.

$\iota_C : C \to \mathbb{R}^3$ $\iota_C^* \alpha = \vec{F}\cdot d\vec{r} = (\alpha_x, \alpha_y, \alpha_z)\cdot \vec{r}'(t)\, dt$

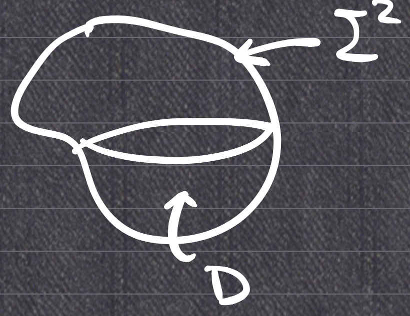
$\iota_\Sigma : \Sigma \to \mathbb{R}^3$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_\Sigma \underbrace{(\nabla\times\vec{F})}_{\leftrightarrow\, d\alpha \ (2\text{-form})}\cdot\hat{n}\, d\sigma$$

$$\oint_C \iota_C^*\alpha = \int_\Sigma \iota_\Sigma^*(d\alpha)$$

$$\oint_{\partial\Sigma} \alpha = \int_\Sigma d\alpha$$

Divergence Theorem:



In 2023:

$$\oint_\Sigma \vec{G}\cdot\hat{n}\, d\sigma = \int_D \nabla\cdot\vec{G}\, \underbrace{dV}_{dx\,dy\,dz}$$

$\vec{G} = \beta_x\hat{i} + \beta_y\hat{j} + \beta_z\hat{k} \leftrightarrow \underset{\substack{\uparrow \\ 2\text{-form}}}{\beta} = \beta_x\, dy\wedge dz + \beta_y\, dz\wedge dx + \beta_z\, dx\wedge dy.$

$\nabla\cdot\vec{G} \leftrightarrow \underbrace{d\beta}_{3\text{-form.}}$

$\vec{G}\cdot\hat{n}\, d\sigma = \iota_\Sigma^*\beta$

$\nabla\cdot\vec{G} \leftrightarrow d\beta = (\nabla\cdot\vec{G})\, dx\wedge dy\wedge dz.$

$$\oint_{\Sigma} \vec{G} \cdot \hat{n}\, d\sigma = \int_{D} \nabla \cdot \vec{G}\, \underbrace{dV}_{dx\,dy\,dz} \qquad (2023)$$

$$\oint_{\Sigma} \iota_{\Sigma}^{*} \beta = \int_{D} d\beta \qquad (4033)$$

$$\boxed{\oint_{\partial D} \beta = \int_{D} d\beta}$$

$$\langle M, \beta \rangle := \int_{M} \beta$$

$$\oint_{\partial M} \beta = \int_{M} d\beta$$

$$\langle \partial M, \beta \rangle = \langle M, d\beta \rangle$$