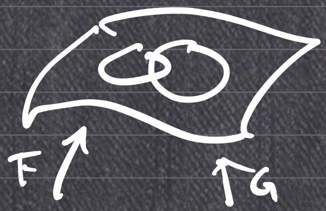
$f: M \to \mathbb{R}$

$$df := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, du^i$$

$F(u_1, \dots, u_n) : \mathcal{U} \to M.$



$$\updownarrow$$

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)$$

du^i

$$\underbrace{\sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \frac{\partial}{\partial u_i}}$$

Exercise: not indep. of local coordinates.

$u^i : \mathcal{U} \subset M \to \mathbb{R}$

$p \mapsto u^i(p).$

$$du^i := \sum_j \frac{\partial u^i}{\partial u_j} \underbrace{du^j}_{\text{cotangent.}} = \sum_j \delta_{ij} \cdot du^j = du^i$$

exterior derivative

1-form: $\alpha = \sum_i \underbrace{\alpha_i}_{\text{scalar}} du^i$

$$d\alpha := \sum_i \underbrace{d\alpha_i}_{\text{1-form}} \wedge du^i \quad \leftarrow \text{2-form}$$

$$= \sum_i \underbrace{\sum_j \frac{\partial \alpha_i}{\partial u_j}}_{d\alpha_i} du^j \wedge du^i$$

Exercise: check $d\alpha$ is indep. of local coordinates.

$$= \left(\sum_{i<j} + \sum_{j<i} \right) \frac{\partial \alpha_i}{\partial u_j} du^j \wedge du^i$$

$$= \sum_{i<j} \frac{\partial \alpha_i}{\partial u_j} du^j \wedge du^i + \sum_{i<j} \frac{\partial \alpha_j}{\partial u_i} du^i \wedge du^j$$

$$= - \sum_{i<j} \frac{\partial \alpha_i}{\partial u_j} du^i \wedge du^j + \sum_{i<j} \frac{\partial \alpha_j}{\partial u_i} du^i \wedge du^j$$

$$= \sum_{i<j} \left(\frac{\partial \alpha_j}{\partial u_i} - \frac{\partial \alpha_i}{\partial u_j} \right) du^i \wedge du^j .$$

In $\mathbb{R}^3$:

$$\alpha = P\,dx + Q\,dy + R\,dz \leftrightarrow \vec{F} = (P, Q, R)$$
$$= P\hat{i} + Q\hat{j} + R\hat{k}.$$

$$d\alpha = d(P\,dx + Q\,dy + R\,dz)$$
$$= dP \wedge dx + dQ \wedge dy + dR \wedge dz$$
$$= \left(\cancel{\frac{\partial P}{\partial x} dx} + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz \right) \wedge dx$$
$$+ \left(\frac{\partial Q}{\partial x} dx + \cancel{\frac{\partial Q}{\partial y} dy} + \frac{\partial Q}{\partial z} dz \right) \wedge dy$$
$$+ \left(\frac{\partial R}{\partial x} dx + \frac{\partial R}{\partial y} dy + \cancel{\frac{\partial R}{\partial z} dz} \right) \wedge dz$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \quad \longleftrightarrow \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \hat{k}$$
$$+ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz \quad \longleftrightarrow + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \hat{i}$$
$$+ \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx \quad \longleftrightarrow + \underbrace{\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)}_{\nabla \times \vec{F}} \hat{j}$$

$$\beta = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$\vec{G} := P\,\hat{i} + Q\,\hat{j} + R\,\hat{k}$$

$$d\beta = dP\wedge dy\wedge dz + dQ\wedge dz\wedge dx + dR\wedge dx\wedge dy$$

$$\boxed{\begin{aligned} \alpha &= \sum \alpha_{i_1\cdots i_k}\, du^{i_1}\wedge\cdots\wedge du^{i_k} \\ \Rightarrow d\alpha &= \sum d\alpha_{i_1\cdots i_k}\wedge du^{i_1}\wedge\cdots\wedge du^{i_k} \end{aligned}}$$

$$= \left(\frac{\partial P}{\partial x}dx + \cancel{\frac{\partial P}{\partial y}dy} + \cancel{\frac{\partial P}{\partial z}dz}\right)\wedge dy\wedge dz$$

$$+ \frac{\partial Q}{\partial y}\,dy\wedge dz\wedge dx + \frac{\partial R}{\partial z}\,dz\wedge dx\wedge dy.$$

$$= \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)}_{\mathrm{div}(\vec{G}) = \nabla\cdot\vec{G}}\, dx\wedge dy\wedge dz.$$

$$\nabla\times\nabla f = 0$$

$$\nabla\cdot(\nabla\times\vec{F}) = 0.$$

$$\begin{tikzcd} \Lambda^k T^*M \arrow[r, "d"] \arrow[rr, bend right, "d\circ d = 0"'] & \Lambda^{k+1} T^*M \arrow[r, "d"] & \Lambda^{k+2} T^*M. \end{tikzcd}$$

Proof of $d^2=0$:

$$\omega = \sum_{i_1,\dots,i_k} \omega_{i_1\dots i_k}\, du^{i_1}\wedge\dots\wedge du^{i_k}$$

$$d\omega = \sum_{i_1,\dots,i_k} d\omega_{i_1\dots i_k}\, du^{i_1}\wedge\dots\wedge du^{i_k}$$

$$= \sum_{i_1,\dots,i_k,j} \frac{\partial \omega_{i_1\dots i_k}}{\partial u_j}\, du^j\wedge du^{i_1}\wedge\dots\wedge du^{i_k}$$

$$d^2\omega = d(d\omega)$$

$$= \sum_{i_1,\dots,i_k,j,l}\left(\underbrace{\frac{\partial}{\partial u_l}\left(\frac{\partial\omega_{i_1\dots i_k}}{\partial u_j}\right)du^l}_{d\left(\frac{\partial\omega_{i_1\dots i_k}}{\partial u_j}\right)}\wedge du^j\right)\wedge du^{i_1}\wedge\dots\wedge du^{i_k}$$

$$= \sum_{i_1,\dots,i_k}\underbrace{\left(\sum_{j<l}+\sum_{l<j}\right)\frac{\partial^2\omega_{i_1\dots i_k}}{\partial u_l\,\partial u_j}\, du^l\wedge du^j}_{\boxed{=0}}\wedge du^{i_1}\wedge\dots\wedge du^{i_k}$$

$$= \sum_{j<l}\underbrace{\frac{\partial^2\omega_{i_1\dots i_k}}{\partial u_l\,\partial u_j}}_{=}\underbrace{du^l\wedge du^j}_{+} + \underbrace{\sum_{j<l}\underbrace{\frac{\partial^2\omega_{i_1\dots i_k}}{\partial u_j\,\partial u_l}}_{=}\underbrace{du^j\wedge du^l}_{-}}_{l\leftrightarrow j}$$

$$= 0.$$

α is a closed form $\overset{\text{def}}{\Longleftrightarrow}$ $d\alpha=0$.

β is a exact form $\overset{\text{def}}{\Longleftrightarrow}$ $\beta = d\eta$. $\exists\eta$

$d^2\alpha = 0 \Rightarrow$ every exact form must be closed.

$\forall \beta$ s.t. $\underset{\text{exact}}{\beta} = d\eta \Rightarrow d\beta = d(d\eta) = d^2\eta = 0.$

$\Rightarrow \beta$ is closed.

But closed form may not be exact.

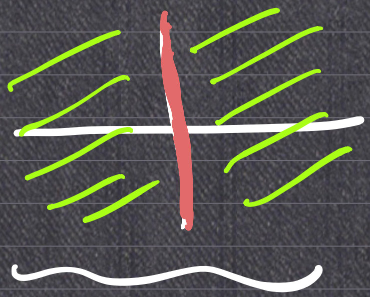
e.g. $\mathbb{R}^2 \setminus \{\vec{0}\}$

$$\omega = -\frac{dx}{x^2+y^2} + \frac{dy}{x^2+y^2}$$

$d\omega \underset{\text{exercise}}{=} 0.$

$$d\left(\tan^{-1}\frac{y}{x}\right) = \omega.$$

on



If $df = \omega$

where $f: \mathbb{R}^2\setminus\{0\} \to \mathbb{R}$.

then $d\left(f - \tan^{-1}\frac{y}{x}\right) = 0$ on $\mathbb{R}^2 \setminus \{y\text{-axis}\}$.

$$\Rightarrow f - \tan^{-1}\frac{y}{x} = \begin{cases} c_1 & \text{if } x>0 \\ c_2 & \text{if } x<0. \end{cases}$$

$$\{\alpha\in\Lambda^k : \alpha \text{ is exact}\} \underset{d^2=0}{\subset} \{\alpha \in \Lambda^k : \alpha \text{ is closed}\}.$$

$$H^k_{dR}(M) := \{\alpha \in \Lambda^k : \alpha \text{ is closed in } M\} \Big/ \{\alpha\in\Lambda^k : \alpha \text{ is exact in } M\}$$

de Rham cohomology group.

(a topological invariant)

M and N are diffeomorphic

$\Rightarrow$ $H^k(M)$ and $H^k(N)$ are isomorphic.

Pull-back of tensors

$$\Phi : M \to N$$
$$p \mapsto \Phi(p).$$

$$\begin{tikzcd} M \arrow[r, "\Phi"] & N \\ u_i \arrow[u, "F"] \arrow[r, "G^{-1}\circ\Phi\circ F"'] & v_\alpha \arrow[u, "G"'] \end{tikzcd}$$

$$\Phi^* : T^*_{\Phi(p)}N \to T^*_pM.$$

$$\Phi^*(dv^\alpha) := \sum_i \frac{\partial v^\alpha}{\partial u_i} du^i$$

$$\underset{\text{1-form.}}{\Phi^*\alpha}$$

$$\Phi^*(\underbrace{\alpha \otimes \beta \otimes \cdots \otimes \gamma}_{(k,0)\text{-tensor.}}) = (\Phi^*\alpha)\otimes(\Phi^*\beta)\otimes\cdots\otimes(\Phi^*\gamma).$$

e.g. $\Sigma^2 \subset \mathbb{R}^3$ regular surface

$$\delta := dx\otimes dx + dy\otimes dy + dz\otimes dz$$
$$\delta(\vec{X},\vec{Y}) = \vec{X}\cdot\vec{Y}.$$

$$\iota : \Sigma^2 \to \mathbb{R}^3$$
$$p \mapsto p.$$

$F(u,v)$

$\iota^*\delta$ ← tensors on Σ^2 (in terms of du, dv).

δ: in terms of dx, dy, dz.

$$\iota^*(dx \otimes dx) = (\iota^* dx) \otimes (\iota^* dx).$$

$$\iota^*(dx) = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$$

$$\begin{tikzcd}
\Sigma^2 \arrow[r, hook, "\iota"] & \mathbb{R}^3 \\
\mathbb{R}^2 \arrow[u, "F(u,v)"] \arrow[r, "\widetilde{id} \circ \iota \circ F"] & \mathbb{R}^3 \arrow[u, "id"']
\end{tikzcd}$$

$$(u,v) \mapsto F(u,v).$$
$$= (x(u,v), y(u,v), z(u,v))$$

$$\iota^*(dx \otimes dx)$$
$$= \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right) \otimes \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right)$$
$$= \left(\frac{\partial x}{\partial u}\right)^2 du \otimes du + \left(\frac{\partial x}{\partial u} \frac{\partial x}{\partial v}\right) du \otimes dv$$
$$+ \left(\frac{\partial x}{\partial v} \cdot \frac{\partial x}{\partial u}\right) dv \otimes du + \left(\frac{\partial x}{\partial v}\right)^2 dv \otimes dv.$$

$$\iota^*(dy \otimes dy)$$
$$= \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right) \otimes \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right) = \cdots$$

$$\iota^*(dz \otimes dz)$$
$$= \left(\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv\right) \otimes \left(\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv\right) = \cdots$$