$T: V \times V \to \mathbb{R}$

$T(e_i, e_j) = T_{ij} \quad \forall i,j \iff T = \sum_{i,j} T_{ij}\, e^i \otimes e^j$

$e^i(e_j) = \delta_{ij}$

$S: V \times V \to V$

$S(e_i, e_j) = \sum_k S_{ij}^{\ k} e_k$

$S = \sum_{i,j,k} S_{ij}^{\ k}\, e^i \otimes e^j \otimes e_k$

(Some authors: $S = \sum_{i,j,k} S_{ij}^{\ k}\, e_k \otimes e^i \otimes e^j$)

$\det : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \to \mathbb{R}$

$\det \begin{bmatrix} \big(\big) & \big(\big) & \cdots & \big(\big) \end{bmatrix}$

<u>$n=2$</u>: $\det(e_1, e_1) = 0 \qquad \det(e_1, e_2) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

$\det(e_2, e_1) = -1 \qquad \det(e_2, e_2) = 0.$

$\det = e^1 \otimes e^2 - e^2 \otimes e^1 = dx \otimes dy - dy \otimes dx.$

$\mathbb{R}^2 \ (x,y)$

cross product.

$\mu : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \mu(v,w) := v \times w$

$\mu(e_1, e_1) = 0 \qquad \mu(e_1, e_2) = e_3 \qquad \mu(e_1, e_3) = -e_2$

$\mu(e_2, e_1) = -e_3 \qquad \mu(e_2, e_2) = 0 \qquad \mu(e_2, e_3) = e_1$

$\mu(e_3, e_1) = e_2 \qquad \mu(e_3, e_2) = -e_1 \qquad \mu(e_3, e_3) = 0$

$$\begin{aligned} \mu = {} & e^1 \otimes e^2 \otimes e_3 - e^2 \otimes e^1 \otimes e_3 \\ & + e^2 \otimes e^3 \otimes e_1 - e^3 \otimes e^2 \otimes e_1 \\ & + e^3 \otimes e^1 \otimes e_2 - e^1 \otimes e^3 \otimes e_2 . \end{aligned}$$

$T, S \in V^*$

$T \wedge S := T \otimes S - S \otimes T$, $\quad S \wedge T = S \otimes T - T \otimes S$

$\in V^*$ $\qquad\qquad\qquad\qquad = -T \wedge S.$

$\det : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. $\quad \det = e^1 \wedge e^2 = dx \wedge dy.$

$\mu = (e^1 \wedge e^2) \otimes e_3 + (e^2 \wedge e^3) \otimes e_1 + (e^3 \wedge e^1) \otimes e_2.$

$\mathrm{span}\{e_i\} = V$, $\quad \mathrm{span}\{e^i\} = V^*.$

$$\boxed{e^i \wedge e^i = 0}$$

$T_1, T_2, T_3 \in V^*$

<u>WANT</u>: $T_1 \wedge T_2 \wedge T_3 =$ linear combinations of $T_i \otimes T_j \otimes T_k$.

such swapping any pair of $\{T_1, T_2, T_3\}$ in $T_1 \wedge T_2 \wedge T_3$ gives $-ve$.

e.g. $T_1 \wedge T_2 \wedge T_3 = -T_2 \wedge T_1 \wedge T_3 = -T_1 \wedge T_3 \wedge T_2.$
$= -T_3 \wedge T_2 \wedge T_1$

$$\begin{array}{r} \text{input} \to \\ \text{output} \to \end{array} \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}_{\substack{\shortparallel \\ (1\,2\,3)}} : \{1,2,3\} \to \{1,2,3\}.$$

$1 \mapsto 2$
$2 \mapsto 3$
$3 \mapsto 1$

$(12) = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 2 & 1 & 3 & 4 & \cdots & n \end{pmatrix}$

transposition.

$(1\,2\,3) = (1\,2)(2\,3)$

$(1\,3\,2) = (1\,3)(3\,2)$

<u>Theorem 3.29</u>

$\forall \sigma \in S_n \leftarrow$ n-th permutation group,

$\exists \tau_1, \dots, \tau_k$ transpositions s.t.

$(1\,2)(1\,2)$
$= id.$

$$\sigma = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k$$

and $\underbrace{(-1)^k}_{\substack{\shortparallel \\ sign(\sigma) \\ sgn}}$ is uniquely determined.

$T_1 \wedge \cdots \wedge T_k \qquad T_i \in V^* = span\{e^i\}_{i=1}^n$

$$:= \sum_{\sigma \in S_k} sgn(\sigma)\, T_{\sigma(1)} \otimes T_{\sigma(2)} \otimes \cdots \otimes T_{\sigma(k)}$$

<u>Claim</u>: τ transposition in S_k,

then $T_{\tau(1)} \wedge T_{\tau(2)} \wedge \cdots \wedge T_{\tau(k)}$

$$= - T_1 \wedge T_2 \wedge \cdots \wedge T_k .$$

<u>Proof</u>: $T_{\tau(1)} \wedge T_{\tau(2)} \wedge \cdots \wedge T_{\tau(k)}$

$$= \sum_{\sigma \in S_k} sgn(\sigma)\, T_{\underbrace{\sigma(\tau(1))}_{\sigma\circ\tau(1)}} \otimes T_{\underbrace{\sigma(\tau(2))}_{\sigma\circ\tau(2)}} \otimes \cdots \otimes T_{\underbrace{\sigma(\tau(k))}_{\sigma\circ\tau(k)}}$$

bijective.
$\underbrace{\sigma \mapsto \sigma\circ\tau}$

$\sigma' = \sigma\circ\tau \Leftrightarrow \sigma = \sigma'\circ\tau$

$$= \sum_{\sigma' \in S_k} \underbrace{\operatorname{sgn}(\sigma' \circ \tau)}_{-\operatorname{sgn}(\sigma')} T_{\sigma'(1)} \otimes T_{\sigma'(2)} \otimes \cdots \otimes T_{\sigma'(k)}$$

$$= - \sum_{\sigma' \in S_k} \operatorname{sgn}(\sigma') T_{\sigma'(1)} \otimes T_{\sigma'(2)} \otimes \cdots \otimes T_{\sigma'(k)}$$

$$= - T_1 \wedge T_2 \wedge \cdots \wedge T_k .$$

$$(T_1 \wedge T_2) \wedge (T_3 \wedge T_4 \wedge T_5) := T_1 \wedge T_2 \wedge T_3 \wedge T_4 \wedge T_5 .$$

$$(-T_2 \wedge T_1) \wedge (T_3 \wedge T_4 \wedge T_5) = - T_2 \wedge T_1 \wedge T_3 \wedge T_4 \wedge T_5$$
$$= T_1 \wedge T_2 \wedge T_3 \wedge T_4 \wedge T_5 .$$

<u>Prop:</u>

$\omega \in \Lambda^k(V^*) := \operatorname{span}\left\{ e^{i_1} \wedge \cdots \wedge e^{i_k} \right\}_{i_1, \ldots, i_k = 1}^{\dim V}$

$\eta \in \Lambda^l(V^*)$

then $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega .$

<u>Proof:</u> $(e^{i_1} \wedge \cdots \wedge e^{i_k}) \wedge (e^{j_1} \wedge \cdots \wedge e^{j_l})$

k-times

k-times.

<u>Total:</u> kl times : $(-1)^{kl}$

(!) $e^i \wedge e^i = 0 .$

$\omega \in \Lambda^2(V^*)$, $\omega \wedge \omega$ is not necessarily 0.

$\omega = e^1 \wedge e^2 + e^3 \wedge e^4$ in $\mathbb{R}^4$.

$$\omega \wedge \omega = (e^1 \wedge e^2 + e^3 \wedge e^4) \wedge (e^1 \wedge e^2 + e^3 \wedge e^4)$$

$$= e^1 \wedge e^2 \wedge e^3 \wedge e^4 + \underbrace{e^3 \wedge e^4 \wedge e^1 \wedge e^2}_{= e^1 \wedge e^2 \wedge e^3 \wedge e^4}$$

$$= 2 e^1 \wedge e^2 \wedge e^3 \wedge e^4 \neq 0.$$

$\eta \in \Lambda^3(V^*)$. $\quad \eta \wedge \eta = (-1)^{3 \times 3} \eta \wedge \eta = -\eta \wedge \eta.$

$$\Rightarrow \eta \wedge \eta = 0 \quad (\operatorname{char} \mathbb{R} \neq 2).$$

$\forall i = 1, 2, \ldots, n$:

$$\omega^i := \sum_{j=1}^{n} a^i_j e^j \qquad V = \operatorname{span}\{e_i\}_{i=1}^{n}$$

$$V^* = \operatorname{span}\{e^j\}_{j=1}^{n}.$$

$\omega^1 \wedge \cdots \wedge \omega^n$

$$= \left(\sum_{j_1=1}^{n} a^1_{j_1} e^{j_1}\right) \wedge \left(\sum_{j_2=1}^{n} a^2_{j_2} e^{j_2}\right) \wedge \cdots \wedge \left(\sum_{j_n=1}^{n} a^n_{j_n} e^{j_n}\right)$$

$$= \sum_{\substack{j_1, \ldots, j_n = 1 \\ \text{distinct}}}^{n} a^1_{j_1} a^2_{j_2} \cdots a^n_{j_n} \, e^{j_1} \wedge e^{j_2} \wedge \cdots \wedge e^{j_n}$$

$$= \sum_{j \in S_n} a^1_{j(1)} a^2_{j(2)} \cdots a^n_{j(n)} \underbrace{e^{j(1)} \wedge e^{j(2)} \wedge \cdots \wedge e^{j(n)}}_{\operatorname{sgn}(j) e^1 \wedge \cdots \wedge e^n}$$

$$= \Big(\underbrace{\sum_{j \in S_n} \operatorname{sgn}(j) \, a^1_{j(1)} a^2_{j(2)} \cdots a^n_{j(n)}}_{\det [a^i_j]} \Big) e^1 \wedge \cdots \wedge e^n.$$

$$\omega^1 \wedge \cdots \wedge \omega^n = \det[a_j^i]\, e^1 \wedge \cdots \wedge e^n .$$

$$T_p^* M = \operatorname{span}\{du^i|_p\}_{i=1}^n = \operatorname{span}\{dv^\alpha|_p\}_{\alpha=1}^n$$

$$du^i = \sum_{\alpha=1}^n \frac{\partial u_i}{\partial v_\alpha}\, dv^\alpha$$

$$du^1 \wedge du^2 \wedge \cdots \wedge du^n = \underbrace{\det\left[\frac{\partial u_i}{\partial v_\alpha}\right]} dv^1 \wedge \cdots \wedge dv^n .$$

$$\left\{ du^1 \wedge \cdots \wedge du^n = \det \frac{\partial(u_1, \ldots, u_n)}{\partial(v_1, \ldots, v_n)}\, dv^1 \wedge \cdots \wedge dv^n \right\}$$

c.f. change of coordinates formula in 2023.

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta \qquad \begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$= \cos\theta\, dr - r\sin\theta\, d\theta$$

$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$dx \wedge dy = (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta)$$

$$= r\cos^2\theta\, \underline{dr \wedge d\theta} - r\sin^2\theta\, \underline{d\theta \wedge dr}$$

$$= \underline{r\cos^2\theta}\, dr \wedge d\theta \underline{+ r\sin^2\theta}\, dr \wedge d\theta$$

$$= r\, dr \wedge d\theta .$$

§ 3.5 Exterior derivatives.

$f : M \to \mathbb{R}$

$$df := \sum_{i=1}^{n} \frac{\partial f}{\partial u_i} du^i$$

Verify indep. of local coordinates.

$$\sum_{i=1}^{n} \frac{\partial f}{\partial u_i} du^i = \sum_{i,\alpha\beta}^{n} \underbrace{\frac{\partial f}{\partial v_\alpha} \frac{\partial v_\alpha}{\partial u_i}}_{\frac{\partial f}{\partial u_i}} \underbrace{\frac{\partial u^i}{\partial v_\beta} dv^\beta}_{du^i}$$

$$= \sum_{\alpha,\beta}^{n} \frac{\partial f}{\partial v_\alpha} \underbrace{\frac{\partial v_\alpha}{\partial v_\beta}}_{\delta_{\alpha\beta}} dv^\beta = \sum_{\alpha} \frac{\partial f}{\partial v_\alpha} dv^\alpha$$

Exercise: $\displaystyle\sum_{i=1}^{n} \frac{\partial f}{\partial u_i} \frac{\partial}{\partial u_i}$ is not indep. of local coordinates.