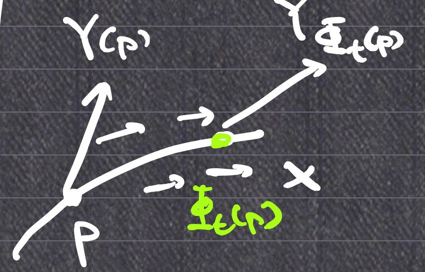
$$(\mathcal{L}_X Y)(p) := \frac{d}{dt}\Big|_{t=0} (\Phi_{-t})_*\left(Y_{\Phi_t(p)}\right)$$

$$= [X, Y]$$



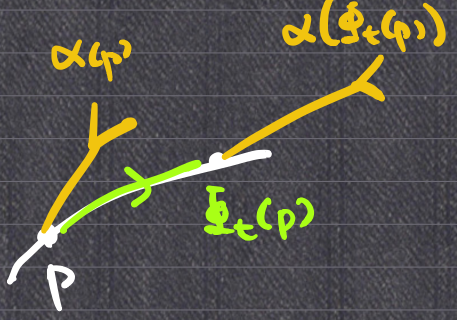
$$\frac{d}{dt}\Phi_t(p) = X(\Phi_t(p))$$

α : 1-form.

$$(\mathcal{L}_X \alpha)(p) := \frac{d}{dt}\Big|_{t=0} \Phi_t^*\left(\alpha_{\Phi_t(p)}\right)$$

↑

vector field



$$F^{-1}\circ\Phi_t\circ F(u_1, \dots, u_n) = (v_t^1, \dots, v_t^n)$$

$$\Phi_t : p \mapsto \Phi_t(p)$$

$$(\Phi_t)^*_p : T^*_{\Phi_t(p)}M \to T^*_p M$$

$$\frac{\partial \Phi_t(p)}{\partial t} = X_{\Phi_t(p)}$$

$$\Leftrightarrow \quad \frac{\partial v_t^i}{\partial t} = X^i_{\Phi_t(p)}.$$

$$\Phi_t^*(\alpha) = \Phi_t^*\left(\sum_i \alpha_i du^i\right) = \sum_i \alpha_i(\Phi_t(p)) \sum_j \frac{\partial v_t^i}{\partial u_j} du^j$$

$$\frac{d}{dt}\Big|_{t=0} \Phi_t^*\alpha = \sum_{i,j} \frac{\partial}{\partial t}\left(\alpha_i(\Phi_t(p)) \frac{\partial v_t^i}{\partial u_j}\right) du^j \Big|_{t=0}.$$

$$= \sum_{i,j}\left(\sum_k \frac{\partial \alpha_i}{\partial u_k} \frac{\partial}{\partial t} v_t^k \cdot \frac{\partial v_t^i}{\partial u_j} + \alpha_i(\Phi_t(p)) \frac{\partial}{\partial t} \frac{\partial v_t^i}{\partial u_j}\right) du^j \Bigg|_{t=0}$$

$$v_0^i = u^i$$

$$= \sum_{i,j}\left(\sum_k \frac{\partial \alpha_i}{\partial u_k} X^k \frac{\partial u^i}{\partial u_j} + \sum_{i,j} \alpha_i|_p \frac{\partial X^i}{\partial u_j}\right) du^j$$

δ_{ij}

$$= \sum_j \sum_k X^k \frac{\partial \alpha_j}{\partial u_k} du^j + \sum_{i,j} \alpha_i(p) \frac{\partial X^i}{\partial u_j} du^j$$

$$= \sum_j \Big(\sum_i \big(X^i \frac{\partial \alpha_j}{\partial u_i} + \alpha_i \frac{\partial X^i}{\partial u_j} \big) \Big) du^j$$

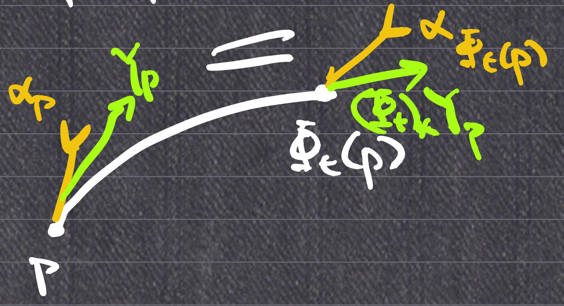
Meaning of $\mathcal{L}_X \alpha \equiv 0$?

$$0 = (\mathcal{L}_X \alpha)(p) = \frac{d}{dt}\Big|_{t=0} (\Phi_t)^* \alpha_{\Phi_t(p)}$$

$$\Rightarrow \quad \Phi_t^* \alpha_{\Phi_t(p)} = \alpha_p \quad \forall t \in \mathbb{R}.$$

$$\Rightarrow \left(\Phi_t^* \alpha_{\Phi_t(p)}\right)(Y_p) = \alpha_p(Y_p) \qquad \forall t \in \mathbb{R},\ \forall Y_p \in T_pM.$$

$$\Rightarrow \quad \alpha_{\Phi_t(p)}\left((\Phi_t)_* Y_p\right) = \alpha_p(Y_p)$$



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§3.3 Tensor product

V, W finite dim'l vector spaces

V^*, W^* dual spaces of V and W.

$l \in V^*, \quad l : V \to \mathbb{R}$

$\alpha \in W^*, \quad \alpha : W \to \mathbb{R}.$

$$l \otimes \alpha : V \times W \to \mathbb{R}$$

$$(l \otimes \alpha)(v, w) := \underbrace{l(v)}_{\in \mathbb{R}} \underbrace{\alpha(w)}_{\in \mathbb{R}}.$$

$V_1, \ldots, V_n$ finite dim vector spaces.

$V_1^*, \ldots, V_n^*$ dual spaces.

$\rotatebox{90}{$\in$}\ l_1 \qquad \rotatebox{90}{$\in$}\ l_n$

$$\operatorname{span}\{ l_1 \otimes \cdots \otimes l_n : V_1 \times \cdots \times V_n \to \mathbb{R} \} =: V_1^* \otimes \cdots \otimes V_n^*$$

$$(l_1 \otimes \cdots \otimes l_n)(v_1, \ldots, v_n) = l_1(v_1) \cdot l_2(v_2) \cdots l_n(v_n).$$

Check:

① $l_1 \otimes c l_2 = c(l_1 \otimes l_2)$

② $(l_1 + \tilde{l}_1) \otimes l_2 = l_1 \otimes l_2 + \tilde{l}_1 \otimes l_2$

③ $(l_1 \otimes l_2)(v_1 + c v_2, w) = (l_1 \otimes l_2)(v_1, w)$
$\qquad + c (l_1 \otimes l_2)(v_2, w)$

$$T_pM = \operatorname{span}\left\{ \frac{\partial}{\partial u_i} \right\}_{i=1}^n, \quad T_p^*M = \operatorname{span}\left\{ du^i|_p \right\}_{i=1}^n.$$

$$du_p^i \otimes du_p^j : T_pM \times T_pM \to \mathbb{R}.$$

$$(du^i \otimes du^j)\left(\frac{\partial}{\partial u_k}, \frac{\partial}{\partial u_l}\right)$$

$$= du^i\left(\frac{\partial}{\partial u_k}\right) \cdot du^j\left(\frac{\partial}{\partial u_l}\right) = \delta_{ik} \cdot \delta_{jl}$$

$$(du^j \otimes du^i)\left(\frac{\partial}{\partial u_k}, \frac{\partial}{\partial u_l}\right) = \delta_{jk}\delta_{il}$$

$B : T_pM \times T_pM \to \mathbb{R}$ bilinear.

$$B_{ij} := B\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$$

$$\Leftrightarrow \quad B = \sum_{i,j} B_{ij}\, du^i \otimes du^j$$

$$\left(\sum_{k,l} B_{kl}\, du^k \otimes du^l\right)\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)$$

$$= \sum_{k,l} B_{kl}\, \underline{(du^k \otimes du^l)}\, \underline{\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right)}$$

$$= \sum_{k,l} B_{kl}\, \delta_{ki}\, \delta_{lj} = B_{ij}$$

Given: $\begin{cases} \omega : T_pM \times T_pM \to \mathbb{R}, \quad \omega \in T_p^*M \otimes T_p^*M. \\ \omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_1}\right) = 0, \quad \omega\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right) = 3 \\ \omega\left(\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_1}\right) = \underline{-3}, \quad \omega\left(\frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_2}\right) = 0. \end{cases}$

$$\omega = 3\, du^1 \otimes du^2 - 3\, du^2 \otimes du^1.$$

$\begin{cases} g \in T_p^*\mathbb{R}^2 \otimes T_p^*\mathbb{R}^2 \\ g := dx \otimes dx + dy \otimes dy. \end{cases}$ $\quad \mathbb{R}^2(x,y), \quad \frac{\partial}{\partial x}, \frac{\partial}{\partial y}.$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta$$
$$= \cos\theta\, dr - r\sin\theta\, d\theta$$

$\frac{\partial}{\partial r} = \cdots \quad \frac{\partial}{\partial \theta}$

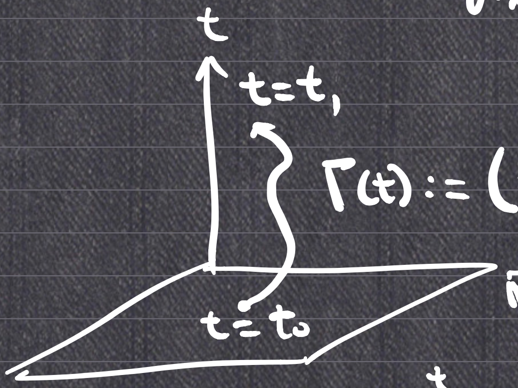
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$

$$\begin{aligned}
g &= dx \otimes dx + dy \otimes dy \\
&= (\cos\theta\, dr - r\sin\theta\, d\theta) \otimes (\cos\theta\, dr - r\sin\theta\, d\theta) \\
&\quad + (\sin\theta\, dr + r\cos\theta\, d\theta) \otimes (\sin\, dr + r\cos\theta\, d\theta) \\
&= \cos^2\theta\, dr\otimes dr \cancel{- r\cos\theta\sin\theta\, dr\otimes d\theta} \cancel{- r\sin\theta\cos\theta\, d\theta\otimes dr} \\
&\quad + r^2\sin^2\theta\, d\theta\otimes d\theta \\
&\quad + \sin^2\theta\, dr\otimes dr \cancel{+ r\cos\theta\sin\theta\, dr\otimes d\theta} \cancel{+ r\cos\theta\sin\theta\, d\theta\otimes dr} \\
&\quad + r^2\cos^2\theta\, d\theta\otimes d\theta \\
&= dr\otimes dr + r^2\, d\theta\otimes d\theta .
\end{aligned}$$

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial\theta}\right) = 0 . \qquad g\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right) = r^2$$

$$\eta = -c^2dt^2 + \underbrace{dx^2}_{\substack{\shortparallel \\ dx\otimes dx}} + dy^2 + dz^2 .$$

$$dx \cdot dy = \tfrac{1}{2}(dx\otimes dy + dy\otimes dx)$$



$$\Gamma(t) := (t, \underbrace{\gamma(t)}_{\in \mathbb{R}^3}) .$$

$$\tau := \int_{t_0}^{t_1} \sqrt{-\eta(\Gamma', \Gamma')}\, dt \qquad \text{proper time.}$$

$$\rho' \qquad = \int_{t_0}^{t_1} \sqrt{-\eta\left(\frac{\partial}{\partial t}+r', \frac{\partial}{\partial t}+r'\right)}\, dt$$

$$= \frac{\partial}{\partial t}+r'(t) \qquad = \int_{t_0}^{t_1} \sqrt{c^2 - \underbrace{(dx^2+dy^2+dz^2)(r', r')}_{=|r'|^2}}\, dt$$

norm in $\mathbb{R}^3$.

$$\leq c(t_1 - t_0)$$

$T: V \to W$ linear map. $\quad V = \operatorname{span}\{e_i\}_{i=1}^{n}$

vector spaces. $\quad W = \operatorname{span}\{f_j\}_{j=1}^{m}$

$T \in V^* \otimes W$.

e.g. $\underset{V^*}{S} \otimes \underset{W}{X}$ $\qquad (S \otimes X)(v) = \underbrace{S(v)}_{\in \mathbb{R}} X$

Claim: $T(e_i) = \sum_j T_i^j f_j \quad \forall i$

$$\Leftrightarrow \quad T = \sum_{i,j} T_i^j e^i \otimes f_j$$

e.g. $T: \mathbb{R}^2 \to \mathbb{R}^2$.

$T(e_1) = (\cos\theta, \sin\theta) = (\cos\theta)e_1 + (\sin\theta)e_2$

$T(e_2) = (-\sin\theta, \cos\theta) = (-\sin\theta)e_1 + (\cos\theta)e_2$

$$T = e^1 \otimes \left((\cos\theta) e_1 + (\sin\theta) e_2 \right)$$
$$+ e^2 \otimes \left(-\sin\theta\, e_1 + \cos\theta \cdot e_2 \right)$$
$$= \cos\theta \cdot e^1 \otimes e_1 + \sin\theta\, e^1 \otimes e_2$$
$$- \sin\theta\, e^2 \otimes e_1 + \cos\theta\, e^2 \otimes e_2 .$$

$e_1 = \frac{\partial}{\partial x}$, $e^1 = dx$

$e_2 = \frac{\partial}{\partial y}$, $e^2 = dy$

$$T = \cos\theta\, dx \otimes \frac{\partial}{\partial x} + \sin\theta\, dx \otimes \frac{\partial}{\partial y} - \sin\theta\, dy \otimes \frac{\partial}{\partial x} + \cos\theta\, dy \otimes \frac{\partial}{\partial y} .$$

$$Rm : T_pM \times T_pM \times T_pM \to T_pM .$$

$$Rm\left(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k} \right) =: \sum_k R_{ijk}^{\ \ \ l} \frac{\partial}{\partial u_l}$$

$$\Leftrightarrow \quad Rm = \sum_{i,j,k,l} R_{ijk}^{\ \ \ l}\, du^i \otimes du^j \otimes du^k \otimes \frac{\partial}{\partial u_l} .$$

$$= \sum R_{ijk}^{\ \ \ l} \frac{\partial u^i}{\partial v_\alpha} dv^\alpha \otimes \frac{\partial u^j}{\partial v_\beta} dv^\beta \otimes \frac{\partial u^k}{\partial v_\gamma} dv^\gamma \otimes \frac{\partial v^\eta}{\partial u_l} \frac{\partial}{\partial v_\eta}$$

$$= \sum \underbrace{R_{ijk}^{\ \ \ l} \frac{\partial u^i}{\partial v_\alpha} \frac{\partial u^j}{\partial v_\beta} \frac{\partial u^k}{\partial v_\gamma} \frac{\partial v^\eta}{\partial u_l}}_{\tilde{R}_{\alpha\beta\gamma}^{\ \ \ \ \eta}} dv^\alpha \otimes dv^\beta \otimes dv^\gamma \otimes \frac{\partial}{\partial v_\eta}$$