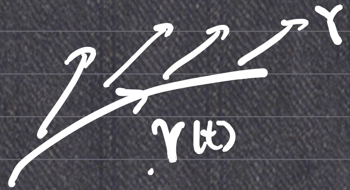
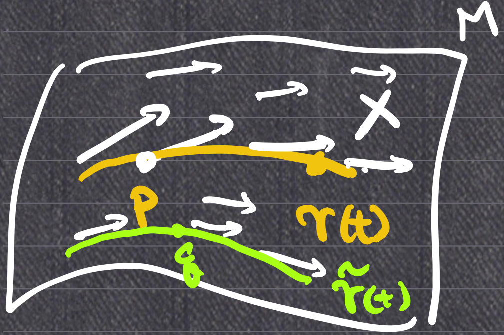


$$\left.\frac{d}{dt}\right|_{t=t_0} Y(t) = \lim_{t\to t_0} \frac{Y(\gamma(t)) - Y(\gamma(t_0))}{t-t_0} \quad ??$$



$\gamma : I \to M$

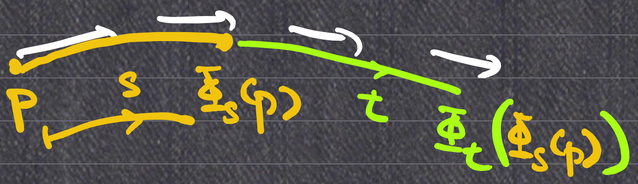
$$\begin{cases} \dfrac{d}{dt}\gamma(t) = X(\gamma(t)) \\ \gamma(0) = p \end{cases} \qquad (*)_p$$

Theory of ODE $\Longrightarrow$ $(*)$ has a unique solution.
(provided that X is C^1).

$\Phi_t(p) :=$ the solution to $(*)_p$.

$$(*)_p \begin{cases} \dfrac{d}{dt}\underbrace{\Phi_t(p)}_{\gamma(t)} = X(\Phi_t(p)) \\ \Phi_0(p) = p. \end{cases}$$

$$\boxed{\Phi_t \circ \Phi_s = \Phi_{t+s}}$$



$= \Phi_{t+s}(p)$.

$$\left.\begin{aligned} \Phi_t \circ \Phi_{-t} &= \Phi_0 = id \\ \Phi_{-t} \circ \Phi_t &= id \end{aligned}\right\} \quad \Phi_{-t} = (\Phi_t)^{-1}$$

$\Phi_t : M \to M$.

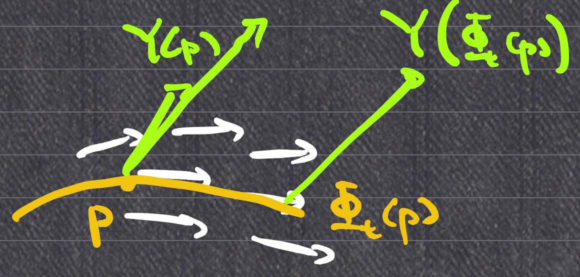
ODE theory $\Longrightarrow$ $\Phi_t : M \to M$ is C^∞ whenever X is C^∞.

$(\Phi_{-t})_*(Y_{\Phi_t(p)})$

$$(\mathcal{L}_X Y)(p) := \frac{d}{dt}\Big|_{t=0} \underbrace{(\Phi_{-t})_*(Y_{\Phi_t(p)})}_{\text{Lie derivative of } Y \text{ along } X}$$

X: direction

Y: want to diff. Y



Prop: $\mathcal{L}_X Y = [X,Y]$.

Def: $[X,Y]f := X(Yf) - Y(Xf)$

$f : M \to \mathbb{R}$

$$\Phi_{-t}(\Phi_t(p)) = p$$

$$(\Phi_{-t})_* : T_{\Phi_t(p)}M \to T_pM.$$

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial u_i}, \quad Y = \sum_{j=1}^n Y^j \frac{\partial}{\partial u_j}$$

$$[X,Y]f = X\Big(\sum_{j=1}^n Y^j \frac{\partial f}{\partial u_j}\Big) - Y\Big(\sum_{i=1}^n X^i \frac{\partial f}{\partial u_i}\Big)$$

$$= \sum_{i=1}^n X^i \frac{\partial}{\partial u_i}\Big(\sum_{j=1}^n Y^j \frac{\partial f}{\partial u_j}\Big) - \sum_{j=1}^n Y^j \frac{\partial}{\partial u_j}\Big(\sum_{i=1}^n X^i \frac{\partial f}{\partial u_i}\Big)$$

$$= \sum_{i=1}^n \sum_{j=1}^n X^i \Big(\frac{\partial Y^j}{\partial u_i}\frac{\partial f}{\partial u_j} + \cancel{Y^j \frac{\partial^2 f}{\partial u_i \partial u_j}}\Big)$$

$$- \sum_{i=1}^n \sum_{j=1}^n Y^j \Big(\frac{\partial X^i}{\partial u_j}\frac{\partial f}{\partial u_i} + \cancel{X^i \frac{\partial^2 f}{\partial u_j \partial u_i}}\Big)$$

$$= \sum_{j=1}^n \Big[\sum_{i=1}^n \Big(X^i \frac{\partial Y^j}{\partial u_i} - Y^i \frac{\partial X^j}{\partial u_i}\Big)\Big] \frac{\partial f}{\partial u_j}$$

$$\Rightarrow [X,Y] = \sum_{j=1}^n \sum_{i=1}^n \Big(X^i \frac{\partial Y^j}{\partial u_i} - Y^i \frac{\partial X^j}{\partial u_i}\Big)\frac{\partial}{\partial u_j}$$

$$\left[x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]f$$

$$= \left(x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\left(x\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\right)$$

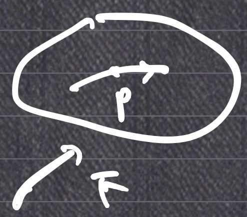
$$= \cancel{x\frac{\partial^2 f}{\partial x^2}} + \cancel{\frac{\partial^2 f}{\partial y\,\partial x}} - \left(\frac{\partial f}{\partial x} + \cancel{x\frac{\partial^2 f}{\partial x^2}} + \cancel{\frac{\partial^2 f}{\partial x\,\partial y}}\right)$$

$$= -\frac{\partial f}{\partial x}$$

$$\Rightarrow \left[x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] = -\frac{\partial}{\partial x}.$$

Claim: $\mathcal{L}_X Y := \frac{d}{dt}\Big|_{t=0}(\Phi_{-t})_*\left(Y_{\Phi_t(p)}\right) = [X,Y].$

$$\begin{tikzcd} M \arrow[r, "\Phi_t"] & M \\ {} \arrow[u, "F"] \arrow[r, "F^{-1}\circ\Phi_t\circ F"'] & {} \arrow[u, "F"'] \end{tikzcd}$$



$$F^{-1}\circ\Phi_t\circ F(u_1,\dots,u_n) = \left(v_t^1(u_1,\dots,u_n),\dots,v_t^n(u_1,\dots,u_n)\right)$$

$$\frac{\partial}{\partial t}\Phi_t(p) = X(\Phi_t(p))$$

$$\sum_{i=1}^n \frac{\partial v_t^i}{\partial t}\frac{\partial}{\partial u_i}\Big|_{\Phi_t(p)} = \sum_{i=1}^n X^i(\Phi_t(p))\frac{\partial}{\partial u_i}\Big|_{\Phi_t(p)}$$

$$\Rightarrow \frac{\partial v_t^i}{\partial t} = X^i(\Phi_t(p)) \quad \forall i.$$

$$(\mathcal{L}_X Y)(p) = \frac{d}{dt}\Big|_{t=0}(\Phi_{-t})_*\left(Y_{\Phi_t(p)}\right)$$

$$= \frac{d}{dt}\Big|_{t=0}(\Phi_{-t})_*\left(\sum_{j=1}^n Y^j(\Phi_t(p))\frac{\partial}{\partial u_j}\Big|_{\Phi_t(p)}\right)$$

$$= \frac{d}{dt}\Big|_{t=0} \sum_{j=1}^{n} Y^j(\Phi_t(p)) \frac{\partial \Phi_{-t}}{\partial u_j}(p)$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial t} Y^j(\Phi_t(p)) \cdot \frac{\partial \Phi_{-t}}{\partial u_j}(p) \Big|_{t=0}$$

$$+ \sum_{j=1}^{n} Y^j(\Phi_t(p)) \frac{\partial}{\partial t}\left(\frac{\partial \Phi_{-t}}{\partial u_j}\right)\Big|_{t=0}.$$

$$= \sum_{j=1}^{n} \frac{\partial}{\partial t} Y^j(\Phi_t(p)) \cdot \frac{\partial}{\partial u_j}(p)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial Y^j}{\partial u_i} \frac{\partial v_t^i}{\partial t} \cdot \frac{\partial}{\partial u_j}(p)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial Y^j}{\partial u_i} X^i \cdot \frac{\partial}{\partial u_j}(p)$$

Y^j
$|$
u_i
$\wedge$
$\dots t$

$$-\sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial X^j}{\partial u_i} Y^i \frac{\partial}{\partial u_j}$$

What does it mean by

$\mathcal{L}_X Y \equiv 0$?

$$(\mathcal{L}_X Y)(p) = \frac{d}{dt}\Big|_{t=0} (\Phi_{-t})_*(Y_{\Phi_t(p)}) = 0 \quad \forall p.$$

$$0 = (\mathcal{L}_X Y)(\Phi_s(p)) = \frac{d}{dt}\Big|_{t=0} (\Phi_{-t})_*(Y_{\Phi_{t+s}(p)})$$

$$= \frac{d}{d(t+s)}\Big|_{t+s=s} (\Phi_{-(t+s)+s})_*(Y_{\Phi_{t+s}(p)})$$

$$= \frac{d}{dt}\Big|_{t=s} (\Phi_{-t+s})_*(Y_{\Phi_t(p)})$$

$\Phi_{-t-s} = \Phi_{-s} \circ \Phi_{-t}$

$$= (\Phi_{+s})_* \left(\frac{d}{dt}\Big|_{t=s} (\Phi_{-t})_*(Y_{\Phi_t(p)})\right.$$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=s} (\Phi_{-t})_*(Y_{\Phi_t(p)}) = 0 \quad \forall s \in \mathbb{R}.$$

$$\Rightarrow (\Phi_{-t})_*(Y_{\Phi_t(p)}) = (\Phi_{-0})_*(Y_{\Phi_0(p)}) = Y(p)$$

$$\forall t \in \mathbb{R}.$$

$$\Rightarrow (\Phi_t)_*^{-1}(Y_{\Phi_t(p)}) = Y(p)$$

$$\Rightarrow \boxed{Y_{\Phi_t(p)} = (\Phi_t)_*(Y_p) \quad \forall t \in \mathbb{R}.}$$

$$\frac{\partial}{\partial s}\psi_s(p) = Y(\psi_s(p))$$

$$\frac{\partial}{\partial t}\Phi_t(p) = X(\Phi_t(p))$$

Claim: if $\mathcal{L}_X Y = 0$

$$\Rightarrow \psi_s \circ \Phi_t = \Phi_t \circ \psi_s$$

$$\forall t, s \in \mathbb{R}.$$

Proof: $$\frac{\partial}{\partial t}\psi_s \circ \Phi_t(p) = \sum_i \frac{\partial \psi_s}{\partial u_i}\frac{\partial v_t^i}{\partial t}$$

$$= \sum_i \frac{\partial \psi_s}{\partial u_i} X^i$$

$$= (\psi_s)_*\left(\sum_i X^i \frac{\partial}{\partial u_i}\right)$$

$$= (\psi_s)_*(X) \leftarrow$$

$$_{\Phi_t(p)} \quad T_{\psi_s \circ \Phi_t(p)} M$$

$$= (\psi_s)_*(X_{\Phi_t(p)})$$

$$= X_{\psi_s \circ \Phi_t(p)}$$

$$\mathcal{L}_X Y = [X, Y] = 0$$

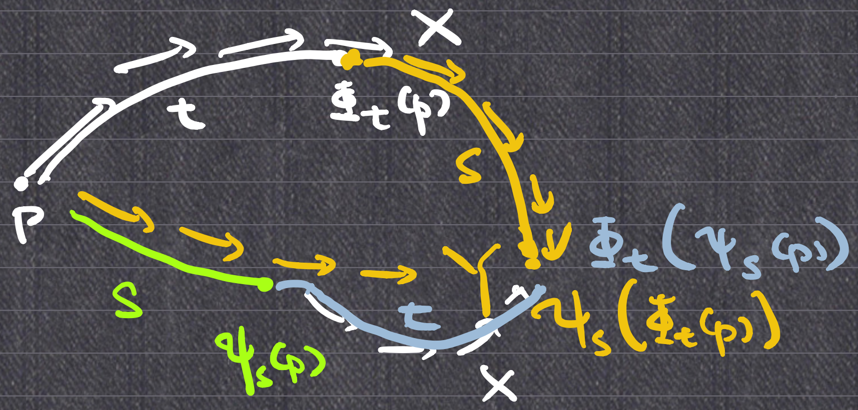
$$\Rightarrow \mathcal{L}_Y X = [Y, X] = 0.$$

$$\underline{X_{\psi_s(p)} = (\psi_s)_*(X_p)}$$

$$\frac{\partial}{\partial t}\Phi_t \circ \psi_s = X(\Phi_t \circ \psi_s(p))$$

Uniqueness Theorem of ODE

$$\Rightarrow \Phi_t \circ \psi_s = \psi_s \circ \Phi_t$$



X
t
$\Phi_t(p)$
s
p
$\Phi_t(\Psi_s(p))$
Y
S
t
$\Psi_s(\Phi_t(p))$
$\Psi_s(p)$
X