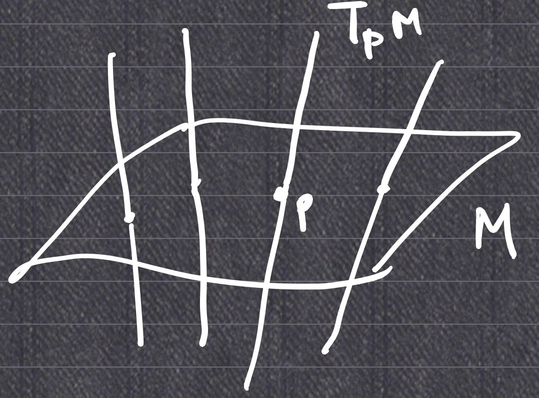
M^n smooth manifold

$TM = \{(p, V_p) : p \in M,\ V_p \in T_pM\}$. tangent bundle

$T^*M = \{(p, \alpha_p) : p \in M,\ \alpha_p \in T_p^*M\}$. cotangent bundle.



tangent bundle is a $2n$-dim C^∞ manifold.

$F(u_1, \ldots, u_n) : U \to O \subset M$

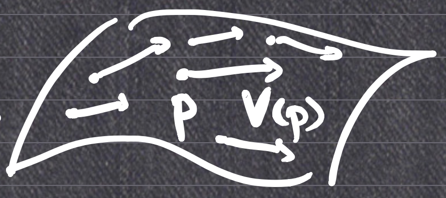
$\downarrow$

$\tilde{F}(u_1, \ldots, u_n, v^1, \ldots, v^n) : U \times \mathbb{R}^n \to TM$

$= \left(F(u_1, \ldots, u_n), \sum_i v^i \frac{\partial}{\partial u_i}\Big|_{F(u_1, \ldots, u_n)}\right)$

Vector field:

$$V : M \longrightarrow TM$$
$$\overset{\in}{p} \longmapsto V(p) \in \{p\} \times T_pM$$



Say V is a C^∞ vector field

$\overset{\text{def}}{\Longleftrightarrow}$ $V : M \to TM$ is C^∞.

$F(u_1, \ldots, u_n) : U \to \underset{\overset{\cup}{p}}{O} \subset M$

$$V(p) = \left(p, \sum_{i=1}^n V^i(p) \frac{\partial}{\partial u_i}(p)\right) \in TM.$$

$\tilde{F}^{-1} \circ V \circ F(u_1, \dots, u_n)$

$= \tilde{F}^{-1}\left(F(u_1, \dots, u_n),\ \sum_{i=1}^{n} V^i(F(u_1, \dots, u_n)) \frac{\partial}{\partial u_i} \right)$

$= \left(u_1, \dots, u_n,\ V^1(F(u_1, \dots, u_n)), \dots, V^n(F(u_1, \dots, u_n)) \right)$

$V: M \to TM$ is C^∞ vector field

$\Leftrightarrow$ components are C^∞.

Differential 1-form

$$\underset{\text{cotangent bundle}}{\alpha : M \to \underbrace{T^*M}} \quad \text{s.t.} \quad \forall p \in M \quad \alpha(p) = (p, \alpha_p), \quad p \in M,\ \alpha_p \in T_p^*M.$$

α is C^∞ 1-form $\Leftrightarrow$ $\alpha : M \to T^*M$ is C^∞.

$$\alpha(p) = (\underbrace{p}_{\in M}, \underbrace{\sum_{i=1}^{n} \alpha_i \, du^i_p}_{\in T_p^*M})$$

Ex: $\alpha : M \to T^*M$ is $C^\infty \Leftrightarrow \alpha_i$ is C^∞ $\forall i$.

V finite dim vector space.

$V^* := \mathrm{Hom}(V, \mathbb{R})$ $\qquad V = \mathrm{span}\{e_\alpha\}_{\alpha=1}^n$

$V^{**} := (V^*)^*$ $\qquad V^* = \mathrm{span}\{e^\alpha\}_{\alpha=1}^n$

$L \in V^{**}$, $\quad L : V^* \to \mathbb{R}$

$V \cong V^{**}$

$v \in V$

$i_v \in V^{**}$ defined as: $i_v(l) := l(v) \in \mathbb{R}$. $\quad l \in V^*$

$v \mapsto i_v$, $\quad V \to V^{**}$ is an isomorphism.

$$i_{v+cw} = i_v + c\, i_w \qquad \forall v, w \in V,\ c \in \mathbb{R}.$$

$$T_pM \longleftrightarrow T_p^*M$$

$$\operatorname{span}\left\{\frac{\partial}{\partial u_i}\right\} \xrightarrow{\ \Phi\ } \operatorname{span}\{du^i\}.$$

$$\frac{\partial}{\partial v_\alpha} \qquad dv^\alpha$$

$$\frac{\partial}{\partial u_i} = \sum_\alpha \frac{\partial v_\alpha}{\partial u_i}\frac{\partial}{\partial v_\alpha}$$

$$du^i = \sum_\alpha \frac{\partial u_i}{\partial v_\alpha} dv^\alpha$$

$$T_pM \longleftrightarrow (T_pM)^{**}$$

$$X \longleftrightarrow i_X \qquad i_X(\alpha) := \alpha(X)$$

Pull-back

$$M \xrightarrow{\Phi} N$$

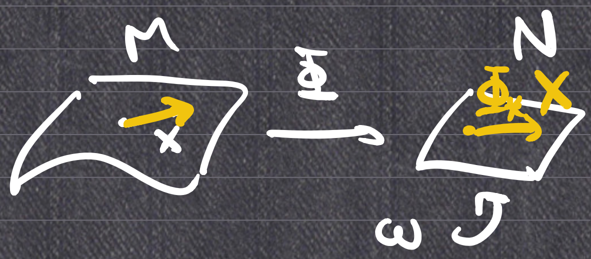
$$T_p^*M \xleftarrow{\Phi^*} T^*_{\Phi(p)}N$$

$$M \xrightarrow{\Phi} N, \quad p \in M$$

$$\Phi_{*p} : T_pM \to T_{\Phi(p)}N.$$

$$\Phi^*(\omega) := ? \in T_p^*M.$$

$$\omega \in T^*_{\Phi(p)}N$$



$$(\Phi^*\omega)(X) := \omega(\underbrace{\Phi_* X}_{(d\Phi)(X)}), \quad X \in T_pM$$

$$\begin{tikzcd} M \arrow[r, "\Phi"] & N \\ \mathbb{R}^m \arrow[u, "F(u_i)"] \arrow[r, "G^{-1}\circ\Phi\circ F"] & \mathbb{R}^n \arrow[u, "G(v_\alpha)"'] \end{tikzcd}$$

$$T_p^*M \xleftarrow{\Phi_p^*} T^*_{\Phi(p)}N$$

$$\Phi^*(dv^\alpha) \overset{?}{=}$$

$$(v_1,\dots,v_n) = G^{-1}\circ\Phi\circ F(u_1,\dots,u_m)$$

$$\begin{aligned}(\Phi^* dv^\alpha)\left(\frac{\partial}{\partial u_i}\right) &:= dv^\alpha\left(\Phi_*\frac{\partial}{\partial u_i}\right) \\ &= dv^\alpha\left(\frac{\partial\Phi}{\partial u_i}\right) \\ &= dv^\alpha\left(\sum_\beta \frac{\partial v_\beta}{\partial u_i}\frac{\partial}{\partial v_\beta}\right) \\ &= \sum_\beta \frac{\partial v_\beta}{\partial u_i}\delta_{\alpha\beta} = \frac{\partial v_\alpha}{\partial u_i}\end{aligned}$$

$$\Phi^* dv^\alpha = \sum_i \frac{\partial v_\alpha}{\partial u_i} du^i$$

e.g. $\Phi(\theta) : \mathbb{R} \to S^1 \subset \mathbb{R}^2$

$$\theta \mapsto (\underbrace{\cos\theta}_{x}, \underbrace{\sin\theta}_{y}).$$

$$\omega := -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$: diff. 1-form on $\mathbb{R}^2$.

$$\Phi^*_\theta \omega = \Phi^*\left(-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)$$

$$= -\frac{\sin\theta}{1}\Phi^* dx + \frac{\cos\theta}{1}\Phi^* dy$$

$$= -\sin\theta \frac{\partial x}{\partial\theta}d\theta + \cos\theta \frac{\partial y}{\partial \theta}d\theta$$

$$= -\sin\theta(-\sin\theta)d\theta + \cos\theta \cdot \cos\theta\, d\theta$$

$$= d\theta.$$

e.g. $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}.$

$$(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$$

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$\psi^*_{(r,\theta)}\omega = \psi^*\left(-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)$$

$$= -\frac{r\sin\theta}{r^2}\psi^* dx + \frac{r\cos\theta}{r^2}\psi^* dy$$

$$= -\frac{\sin\theta}{r}\left(\frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta\right)$$

$$+ \frac{\cos\theta}{r}\left(\frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta\right)$$

$$= -\frac{\sin\theta}{r}\left(\cancel{\cos\theta\, dr} - r\sin\theta\, d\theta\right)$$

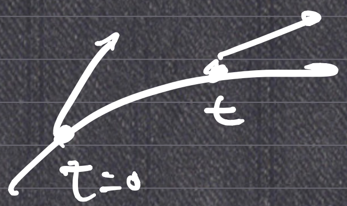
$$+ \frac{\cos\theta}{\cancel{r}}\left(\cancel{\sin\theta\, dr} + r\cos\theta\, d\theta\right)$$

$$= (\sin^2\theta + \cos^2\theta)\, d\theta = d\theta.$$

Lie derivatives

$$\frac{dY}{dt}\Big|_{t=0} = \lim_{t\to 0} \cancel{\frac{Y(\gamma(t)) - Y(\gamma(0))}{t}}$$

$$\gamma : I \to M.$$



$$Y_p = \sum Y^i(p) \frac{\partial}{\partial u_i}(p)$$

$$Y_q = \sum Y^i(q) \frac{\partial}{\partial u_i}(q)$$

$$Y_p - Y_q = \sum_i \left(Y^i(p) - Y^i(q)\right) \frac{\partial}{\partial u_i}(?)$$

$$Y_p = \sum \frac{\partial v_\alpha}{\partial u_i}\Big|_p Y^i(p) \frac{\partial}{\partial v_\alpha}(p), \quad Y_q = \sum \frac{\partial v_\alpha}{\partial u_i}\Big|_q Y^i(q) \frac{\partial}{\partial v_\alpha}(q)$$