香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

FINAL EXAMINATION

Course Code: MATH 4033
Course Title: Calculus on Manifolds
Semester: Spring 2017-18
Date and Time: 28 May 2018, 12:30PM-3:30PM

Instructions

- Do **NOT** open the exam until instructed to do so.
- All mobile phones and communication devices should be switched **OFF**.
- It is an **OPEN-NOTES** exam. Authorized reference materials are the instructor's lecture notes and homework solutions. No other reference materials are allowed.
- Answer **ALL FOUR** problems. Write your solutions in the yellow book.
- You must **SHOW YOUR WORK** to receive credits in every problem, unless otherwise is stated.
- Some problems are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

"I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

Student's Signature: ______________________________

Student's Name: ______________________ **HKUST ID:** __________

Answer **ALL FOUR** problems.
Recommended timing: Q1 $<$ 20min, Q2 $<$ 20min, Q3 $<$ 60min, Q4 $<$ 80min

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1. Consider the following differential forms on $\mathbb{R}^3$ with standard coordinates (x, y, z):

$$\alpha := xe^{z^2-2z}\,dx + (\sin(xyz) + y + 1)\,dy + e^{z^2}\sin(z^2)\,dz$$
$$\beta := -xy\cos(xyz)\,dx + 2xe^{z^2-2z}(z-1)\,dy + yz\cos(xyz)\,dz$$
$$\omega := xe^{z^2-2z}\,dy\wedge dz + (\sin(xyz) + y + 1)\,dz\wedge dx + e^{z^2}\sin(z^2)\,dx\wedge dy$$
$$\eta := -xy\cos(xyz)\,dy\wedge dz + 2xe^{z^2-2z}(z-1)\,dz\wedge dx + yz\cos(xyz)\,dx\wedge dy$$

(a) Some *warm-up* (i.e. stupid) questions: **[6]**

i. Which of the above is/are 1-form(s)? Which is/are 2-form(s)?

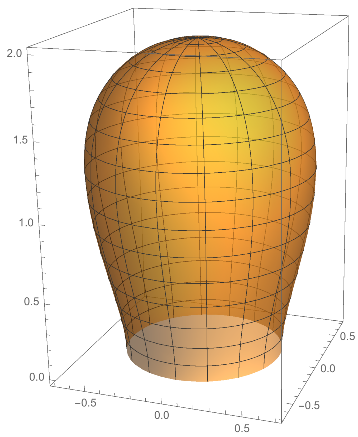
ii. The wedge product of two of the above differential forms is zero. Which two? Explain briefly.

iii. It is known that exactly ONE of the following is true. Which one? Do not need to explain.

$d\alpha = \alpha$	$d\alpha = \beta$	$d\alpha = \omega$	$d\alpha = \eta$
$d\beta = \alpha$	$d\beta = \beta$	$d\beta = \omega$	$d\beta = \eta$
$d\omega = \alpha$	$d\omega = \beta$	$d\omega = \omega$	$d\omega = \eta$
$d\eta = \alpha$	$d\eta = \beta$	$d\eta = \omega$	$d\eta = \eta$

iv. It is known that one of the forms α, β, ω, η is closed. Which one? Explain briefly.

(b) Given Σ is an orientable regular surface in $\mathbb{R}^3$ which stands above the xy-plane with boundary curve $\partial\Sigma$ given by the unit circle $x^2 + y^2 = 1$ on the xy-plane. See the diagram below as a reference. Note that the open ball $x^2 + y^2 < 1$ on the xy-plane is NOT a part of Σ. **[10]**



Let $\iota_\Sigma : \Sigma \to \mathbb{R}^3$ be the inclusion map. Calculate ONE of the following.

$$\int_\Sigma \iota_\Sigma^*\omega \qquad\qquad \int_\Sigma \iota_\Sigma^*\eta$$

State clearly which one you calculate. No bonus for calculating both.
[Remark: You can pick your preferred orientation, but it needs to be consistent throughout your solution.]

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2. Let g be a C^∞ 2-tensor and X be a C^∞ vector field on a C^∞ manifold with local coordinate expressions: **[16]**

$$g = \sum_{i,j} g_{ij}\, du^i \otimes du^j \qquad\qquad X = \sum_k X^k \frac{\partial}{\partial u_k}.$$

Show that the Lie derivative $\mathcal{L}_X g$ has the following local coordinate expression:

$$\mathcal{L}_X g = \sum_{i,j,k} \left(X^k \frac{\partial g_{ij}}{\partial u_k} + g_{kj} \frac{\partial X^k}{\partial u_i} + g_{ik} \frac{\partial X^k}{\partial u_j} \right) du^i \otimes du^j.$$

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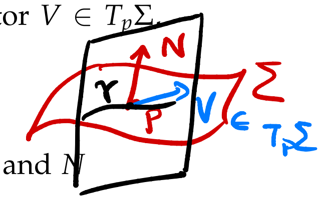
3. Consider a C^∞ function $f : \mathbb{R}^3 \to \mathbb{R}$ whose level set $\Sigma := f^{-1}(0)$ is non-empty and that $\nabla f(q) \neq 0$ for any $q \in \Sigma$. Given a fixed $p \in \Sigma$ and a non-zero tangent vector $V \in T_p\Sigma$, we denote



$$N := \text{a non-zero normal vector to } \Sigma \text{ at } p$$
$$\Pi_p(V,N) := \text{the plane in } \mathbb{R}^3 \text{ passing through } p \text{ and parallel to both } V \text{ and } N$$
$$\gamma := \Pi_p(V,N) \cap \Sigma$$

(a) Find a C^∞ map $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ such that $\gamma = \Phi^{-1}((0,0))$. **[6]**

(b) Show that the set γ is locally a smooth manifold **near** p, i.e. there exists an open set $\mathcal{U}$ in $\mathbb{R}^3$ containing p such that $\gamma \cap \mathcal{U}$ is a smooth manifold. What is its dimension? **[16]**

(c) Show that $\gamma \cap \mathcal{U}$ is a submanifold of Σ. **[8]**

[Hint to all parts: It may be helpful to sketch a diagram first.]

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4. Let $\mathbb{RP}^N$ be the N-th dimensional real projective space.

(a) Show that the following atlas of $\mathbb{RP}^{2n-1}$ (where $n \in \mathbb{N}$) is oriented: **[12]**

$$\mathcal{A} = \{F_i : \mathbb{R}^{2n-1} \to \mathbb{RP}^{2n-1}\}_{i=1}^{2n}$$

$$F_i(u_1, \cdots, u_{i-1}, u_{i+1}, \cdots, u_{2n}) = [u_1 : \cdots : u_{i-1} : (-1)^i : u_{i+1} : \cdots : u_{2n}].$$

(b) Given the following fact: **[6]**

"Any n-form Ω on a compact, connected, orientable n-manifold M^n without boundary, such that $\int_M \Omega = 0$, must be exact."

Using the above fact, show that $b_n(M^n) = 1$ for any such manifold M^n.

(c) Hence, show that: **[20]**

$$b_1(\mathbb{RP}^3) = b_1(\mathbb{RP}^2) \quad \text{and} \quad b_2(\mathbb{RP}^3) = b_2(\mathbb{RP}^2).$$

Justify every claim with at least a brief reason.

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$$\begin{aligned} N &:= \text{a non-zero normal vector to } \Sigma \text{ at } p \\ \Pi_p(V, N) &:= \text{the plane in } \mathbb{R}^3 \text{ passing through } p \text{ and parallel to both } V \text{ and } N \\ \gamma &:= \Pi_p(V, N) \cap \Sigma \end{aligned}$$

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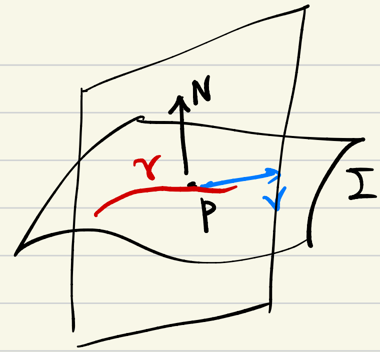

$\Sigma = f^{-1}(0)$

$\nabla f(q) \neq 0 \quad \forall q \in \Sigma.$

(a) <u>WANT</u>: $\gamma = \Phi^{-1}((0,0))$

$$\begin{cases} f(x,y,z) = 0 \\ \underbrace{(N \times V)}_{\text{normal to } \square} \cdot ((x,y,z) - \vec{p}) = 0. \end{cases}$$

$\Phi : \mathbb{R}^3 \to \mathbb{R}^2$, $\Phi(x,y,z) := \left(f(x,y,z), (N\times V)\cdot((x,y,z)-\vec{p})\right)$

then $\Phi^{-1}((0,0)) = \gamma = \Pi_p(V,N) \cap \Sigma$.

<u>Claim</u>: Φ is a submersion at p (hence near p by continuity)

<u>Proof</u>,

$$D\Phi = \begin{array}{c} \\ \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ (N\times V)_x & (N\times V)_y & (N\times V)_z \end{bmatrix} \\ \uparrow \quad\quad \uparrow \quad\quad \uparrow \\ \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z} \end{array} \begin{array}{l} \leftarrow \nabla f(p) \\ \leftarrow N \times V. \end{array}$$

$\{\nabla f(p), N\times V\}$ linearly indep.

(row) rank $= 2$

$\underbrace{N \times V}_{\neq 0}$

$\nabla f(p) // N$

$\therefore$ $D\Phi$ is surjective.

$\underbrace{\vec{N}\times\vec{V}}_{\neq 0} \perp \underbrace{\nabla f(p)}_{\neq 0}.$

$\gamma \cap \mathcal{U} = \Phi^{-1}((0,0)) \cap \mathcal{U}$ is a submanifold of $\mathbb{R}^3$ (dim $= 3-2 = 1$)

(c) $\iota_1 : \Upsilon \cap \mathcal{U} \to \mathbb{R}^3$ ✓ immersion by (b)

<u>Need</u>: $\iota_2 : \Upsilon \cap \mathcal{U} \to \Sigma = f^{-1}(0)$ immersion

$$\begin{tikzcd} \Upsilon \cap \mathcal{U} \arrow[r, "\iota_2"] \arrow[rr, bend right, "\iota_1"'] & \Sigma \arrow[r, "\iota_3"] & \mathbb{R}^3 \end{tikzcd}$$

$\Sigma \subset \mathbb{R}^3$
$\shortparallel$
$f^{-1}(0)$

$\nabla f \neq 0$ on Σ.

$\Rightarrow \Sigma$ regular surface.

$\Rightarrow \iota_3 : \Sigma \to \mathbb{R}^3$ is an immersion ($\Leftarrow$ (3))

$$(\iota_1)_* = (\iota_3 \circ \iota_2)_*$$
$$= (\iota_3)_* (\iota_2)_*$$

✓ ✓

$X \in \ker(\iota_2)_* \Rightarrow (\iota_2)_* X = 0$

$\Rightarrow (\iota_3)_* (\iota_2)_* X = 0$

$\Rightarrow (\iota_1)_* X = 0$

$\Rightarrow X = 0$ ($(\iota_1)_*$ is injective).

$\therefore (\iota_2)_*$ is injective.

$M_{n\times n}(\mathbb{R})$ = set of real $n\times n$ matrix
$\cong \mathbb{R}^{n^2}$

$\mathrm{Sym}_n(\mathbb{R})$ = set of symmetric $n\times n$ real matrices.

$\subset M_{n\times n}(\mathbb{R})$.

$$F(x_{ij}) = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{12} & x_{22} & & \vdots \\ x_{1n} & \cdots & \cdots & x_{nn} \end{bmatrix}.$$

$1 \le i \le j \le n$

$F : \mathbb{R}^{C^n_2} \to \mathrm{Sym}_n(\mathbb{R})$

$$\mathrm{Sym}_n(\mathbb{R}) \xrightarrow{\iota} M_{n\times n}(\mathbb{R})$$

$$\iota_* : T\,\mathrm{Sym}_n(\mathbb{R}) \to \underline{T M_{n\times n}(\mathbb{R})} = \mathbb{R}^{n^2}.$$

$$= \mathrm{span}\left\{ \frac{\partial}{\partial x_{ij}} \right\}_{1\le i\le j\le n}$$

$$\Big\{ \underbrace{\iota_*\Big(\frac{\partial}{\partial x_{ij}}\Big)}_{\parallel} \Big\}_{i\le j} \quad \text{is linearly indep. set of vectors.}$$

$$\frac{\partial \iota}{\partial x_{ij}} = \frac{\partial}{\partial x_{ij}}(\iota \circ F) = \begin{cases} \begin{array}{c|c} & i \\ j & \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix} \end{array} & i=j \\ \begin{array}{c|cc} & & j \\ i & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ j & \end{array} & i<j \end{cases}$$

$$\left\{ \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \end{bmatrix} \cdots \right\} \quad \text{linearly indep.}$$

ι_* is injective $\Rightarrow$ ι is an immersion

$\therefore$ $Sym_n(\mathbb{R})$ is a submanifold of $M_{n\times n}(\mathbb{R})$.

$$O(n) = \{A \in M_{n\times n}(\mathbb{R}) : \underbrace{A^TA = I}\} = \Phi^{-1}(I)$$

$f^{-1}(1)$

$f(x,y,z) = x^2+y^2+z^2$

$f : \mathbb{R}^3 \to \mathbb{R}$

$$\Phi : M_{n\times n}(\mathbb{R}) \longrightarrow Sym_n(\mathbb{R})$$

$$A \longmapsto A^TA$$

WANT: Φ_{*A_0} is surjective $\forall A_0 \in O(n) = \Phi^{-1}(I)$

$\forall B \in Sym_n(\mathbb{R})$, need $C \in M_{n\times n}(\mathbb{R})$ s.t.

$$(\Phi_{*A_0})(C) = B.$$

$$\begin{aligned}(\Phi_{*A_0})\Big(\frac{\partial}{\partial x_{ij}}\Big) &= \frac{\partial \Phi}{\partial x_{ij}}\Big|_{A_0} \\ &= \frac{\partial}{\partial x_{ij}}(A^TA)\Big|_{A_0} \\ &= (E_{ij}^TA + A^TE_{ij})\Big|_{A_0} \\ &= E_{ij}^TA_0 + A_0^TE_{ij}\end{aligned}$$

$$A = [x_{ij}]$$

$$\frac{\partial}{\partial x_{ij}}A = \begin{array}{c|c} & i \\ j & \begin{bmatrix} 0 & & \\ & 1 & \\ & & \end{bmatrix} \end{array}$$

$$E_{ij} = \begin{array}{cc} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & i \\ \quad\ \ j & \end{array}$$

(i, j)-entry of C

$$\left(\Phi_*\right)_{A_0}(C) = \left(\Phi_*\right)_{A_0}\left(\sum_{i,j=1}^{n} c_{ij} \underbrace{E_{ij}}_{= \frac{\partial}{\partial x_{ij}}}\right)$$

$$= \sum_{i,j=1}^{n} c_{ij}\left(E_{ij}^{T} A_0 + A_0^{T} E_{ij}\right)$$

$$= \sum_{i,j=1}^{n} c_{ij}\left(A_0^{T} E_{ij}\right)^{T} + \sum_{i,j=1}^{n} A_0^{T} c_{ij} E_{ij}$$

$$= \sum_{i,j}^{n} c_{ij} E_{ji} A_0 + A_0^{T} C = C^{T} A_0^{T} + A_0^{T} C = \left(A_0^{T} C\right)^{T} + A_0^{T} C$$

& Choose C s.t. $A_0^{T} C = \frac{1}{2} B \iff \boxed{C = \frac{1}{2} A_0 B}$

$$\Longrightarrow \left(\Phi_*\right)_{A_0}(C) = \frac{1}{2}\underbrace{B^{T}}_{= B} + \frac{1}{2} B = B.$$