$\Phi : M \to N$ is a submersion on $\Sigma = \Phi^{-1}(q)$

then $\Phi^{-1}(q)$ is a smooth submfd of M

$(\underbrace{0,\dots,0}_{n}, u_{n+1},\dots u_m)$ and $\dim \Phi^{-1}(q) = \dim M - \dim N$.

Recall Submersion Theorem

$\Rightarrow \ \exists\, F(u_1,\dots,u_m) : \mathcal{U} \to \overset{\ni p}{\mathcal{O}} \subset M$

$G(v_1,\dots,v_n) : \tilde{\mathcal{U}} \to \underset{\ni\, \Phi(p)=q}{\mathcal{S}} \subset N$

s.t.

$$G^{-1}\circ \Phi \circ F(u_1,\dots,u_m) = (u_1,\dots,u_n)$$

$$\tilde{F}(u_{n+1},\dots,u_m) := F(0,\dots,\underset{u_n}{\overset{\uparrow}{0}}, u_{n+1},\dots,u_m)$$

parametrizes Σ near p.

Exercise: $\tilde{F}_1^{-1}\circ \tilde{F}_2 = F_1^{-1}\circ F_2 \big|_{\{0\}\times \mathbb{R}^{m-n}}$.

$\therefore$ Σ is a smooth manifold with $\dim = m-n$.

$\iota : \Sigma \to M$ inclusion.

$$\begin{array}{ccc} \tilde{F}\uparrow & & \uparrow F \\ & \xrightarrow{\;F^{-1}\circ\iota\circ\tilde{F}\;} & \end{array}$$

$$F^{-1}\circ\iota\circ\tilde{F}(u_{n+1},\dots,u_m) = F^{-1}\cancel{\circ\iota}\big(F(0,\dots,0,u_{n+1},\dots u_m)\big)$$

$$= (0,\dots,0,u_{n+1},\dots,u_m)$$

$$D(F^{-1}\circ\iota\circ\tilde{F}) = \underbrace{\left[\begin{array}{c} 0 \\ \hline I \end{array}\right]}_{\text{cols are linearly indep.}}$$

$\therefore$ ι is an immersion.

$\Sigma \subset M$ is a C^∞ submfd of M.

Chapter 3

V = finite dim'l vector space $/\mathbb{R}$.

$= \text{span}\{e_i\}_{i=1}^{n}$

V^* = set of linear maps from V to $\mathbb{R}$.

$= \mathcal{L}(V,\mathbb{R}) = \text{Hom}(V,\mathbb{R})$. $\quad (\dim V^* = \dim V)$

$l \in V^*$, $\quad l : V \to \mathbb{R}$. $\quad l(e_i) \in \mathbb{R}$.

$e^i \in V^*$, $e^i : V \to \mathbb{R}$

$$e^i(e_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} = \delta_{ij}$$

Exercise: $\underbrace{\{e^i\}_{i=1}^n}_{\text{dual basis of } V^* \text{ wrt. } \{e_i\}_{i=1}^n}$ are linearly indep.

$l \in V^*$, $\quad l(e_i) = a_i \quad \forall i = 1, 2, \dots, n.$

Then: $\underbrace{l = \sum_{i=1}^n a_i e^i}$

$$l(e_j) = \sum_{i=1}^n a_i e^i(e_j) = \sum_{i=1}^n a_i \delta_{ij} = a_j .$$

$\therefore \quad l(e_i) = a_i \in \mathbb{R} \iff l = \sum_{i=1}^n a_i e^i$

bra ket.

$\langle e_i|$ $\quad |e_j\rangle$.

$$\langle e_i | e_j \rangle = \delta_{ij}$$

$$\langle T | e_j \rangle = a_j$$

$$\iff \langle T | = \sum_i a_i \langle e_i | .$$

$l \in V^*$ $\quad V = \mathbb{R}^3 = \text{span}\{e_1, e_2, e_3\}$

$$\begin{cases} l(e_1) = 3 \\ l(e_2) = 2 \\ l(e_3) = -5 \end{cases} \iff l = 3e^1 + 2e^2 - 5e^3 .$$

$T_pM = \text{span}\left\{\frac{\partial}{\partial u_i}(p)\right\}$ $\qquad F(u_1,\dots,u_n): U \to O \subset M.$

p.

$\underbrace{T_p^*M}_{} := (T_pM)^*$

cotangent space at p.

Denote $\{du^i|_p\}_{i=1}^n$ to be the dual basis

of T_p^*M wrt $\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^n$

$du^i|_p : T_pM \to \mathbb{R}.$

$du^i|_p\left(\frac{\partial}{\partial u_j}(p)\right) := \delta_{ij}$

$\alpha \in T_p^*M, \quad \alpha: T_pM \to \mathbb{R}.$

$\alpha\left(\frac{\partial}{\partial u_i}\right) = c_i \quad \forall i$

$\alpha = \sum_{i=1}^n c_i\, du^i$

M

$F(u_1,\dots,u_n) \longrightarrow \text{span}\left\{\frac{\partial}{\partial u_i}(p)\right\} = T_pM \longrightarrow \text{span}\{du^i|_p\}$

$= T_p^*M$

$G(v_1,\dots,v_n)$

$\searrow \text{span}\left\{\frac{\partial}{\partial v_j}(p)\right\} = T_pM \rightsquigarrow \text{span}\{dv^j|_p\} = T_p^*M$

$\frac{\partial}{\partial u_i} = \sum_j \frac{\partial v_j}{\partial u_i}\frac{\partial}{\partial v_j}$

$$du^i\left(\frac{\partial}{\partial v_j}\right) = du^i\left(\sum_k \frac{\partial u_k}{\partial v_j}\frac{\partial}{\partial u_k}\right)$$

$$= \sum_k \frac{\partial u_k}{\partial v_j}\delta_{ik} = \frac{\partial u_i}{\partial v_j}$$

$$\Rightarrow \quad du^i = \sum_j \frac{\partial u_i}{\partial v_j} dv^j \quad \leftarrow$$

e.g. $\mathbb{R}^2$ (x,y), (r,θ)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} dx = \dfrac{\partial x}{\partial r}dr + \dfrac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta. \\ dy = \dfrac{\partial y}{\partial r}dr + \dfrac{\partial y}{\partial \theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta. \end{cases}$$

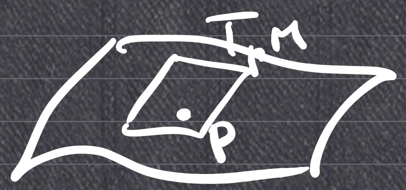
Tangent bundle

informally: union of all tangent spaces of a manifold.

$$TM := \{(p, V_p) : p \in M,\ V_p \in T_pM\}.$$

tangent bundle

Not $M \times TM$



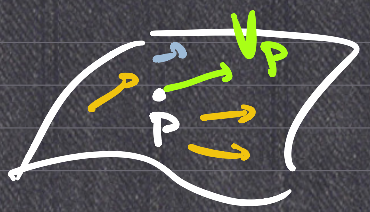
$$\boxed{\begin{aligned} &X \times Y \\ &= \{(x,y) : x \in X,\ y \in Y\} \end{aligned}}$$

Cotangent bundle:

$$T^*M := \{(p, \alpha_p) : p \in M,\ \alpha_p \in T_p^*M\}.$$

Vector field : $V : M \to TM$

$$\underbrace{V(p)}_{(p,\ V_p)} \in T_pM$$



$\underline{\text{Claim}}$: TM is a C^∞-manifold of dimension $= 2\underbrace{\dim M}_{=n}$.

$\underline{\text{Proof}}$: $F(u_1, \dots, u_n) : \mathcal{U} \to \mathcal{O} \subset M$. ($\mathcal{U} \subset \mathbb{R}^n$)

$$\tilde{F}(u_1, \dots, u_n, X^1, \dots, X^n) : \underbrace{\mathcal{U} \times \mathbb{R}^n}_{\subset \mathbb{R}^{2n}} \to TM.$$

$$:= \Big(\underbrace{F(u_1, \dots, u_n)}_{\in M}, \underbrace{\sum_{i=1}^n X^i \frac{\partial}{\partial u_i}\Big|_{F(u_1, \dots, u_n)}}_{\in T_{F(u_1, \dots, u_n)}M} \Big).$$

$G(v_1, \dots, v_n) : \tilde{\mathcal{U}} \to \tilde{\mathcal{O}} \subset M$.

$$\rightsquigarrow \tilde{G}(v_1, \dots, v_n, Y^1, \dots, Y^n) := \Big(G(v_1, \dots, v_n), \sum_{j=1}^n Y^j \frac{\partial}{\partial v_j}\Big|_{\dots} \Big)$$

$$\tilde{G}^{-1} \circ \tilde{F}(u_1, \dots, u_n, X^1, \dots, X^n)$$

$$= \tilde{G}^{-1} \Big(F(u_1, \dots, u_n), \sum_{i=1}^n X^i \frac{\partial}{\partial u_i}\Big|_{F(u_1, \dots, u_n)} \Big)$$

$$= \tilde{G}^{-1} \Big(F(u_1, \dots, u_n), \sum_{i=1}^n \sum_{j=1}^n X^i \frac{\partial v_j}{\partial u_i} \frac{\partial}{\partial v_j} \Big)$$

$$= \Big(G^{-1} \circ F(u_1, \dots, u_n), \sum_{i=1}^n X^i \underbrace{\frac{\partial v_1}{\partial u_i}}_{C^\infty}, \sum_{i=1}^n X^i \underbrace{\frac{\partial v_2}{\partial u_i}}_{C^\infty}, \dots, \sum_{i=1}^n X^i \underbrace{\frac{\partial v_n}{\partial u_i}}_{C^\infty} \Big)$$