$$\Phi : \underset{\substack{\cup\\ p}}{M} \to N$$

$\Phi$ is an immersion at $p$ $\overset{\text{def}}{\iff}$ $\Phi_{*p} : T_pM \to T_{\Phi(p)}N$ is injective.

$\Phi$ is a submersion at $p$ $\overset{\text{def}}{\iff}$ $\Phi_{*p}$ is surjective.

<u>e.g.</u> $f : \mathbb{R}^n \to \mathbb{R}$

$$f_{*p} : \underbrace{T_p\mathbb{R}^n}_{=\mathbb{R}^n} \to \mathbb{R}$$

$$\mathbb{R}^n = \operatorname{span}\{e_i\}_{i=1}^n$$

$$[f_{*p}] = \underbrace{\left[\frac{\partial f}{\partial x_1}(p) \cdots \frac{\partial f}{\partial x_n}(p)\right]}_{n} 1 \qquad T_p\mathbb{R}^n = \operatorname{span}\left\{\frac{\partial}{\partial x_i}(p)\right\}_{i=1}^n$$

$f$ is a submersion at $p$ $\iff$ $\left(\frac{\partial f}{\partial x_1}(p), \ldots, \frac{\partial f}{\partial x_n}(p)\right) \neq (0, \ldots, 0)$

$\iff \nabla f(p) \neq 0$

<u>e.g.</u>

$$\Phi : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{RP}^n.$$

$$\Phi(x_0, \ldots, x_n) := [x_0 : \cdots : x_n].$$

$$\cancel{\Phi([x_0 : \cdots : x_n]) = (x_0, \ldots, x_n)}$$

$$\begin{tikzcd}
 & \mathbb{R}^{n+1}\setminus\{0\} \arrow[r, "\Phi"] & \mathbb{RP}^n & \\
\mathbb{R}^{n+1}\setminus\{0\} \arrow[ur, "id"] \arrow[rr, "F^{-1}\circ\Phi\circ id"'] & & \mathbb{R}^n \arrow[u, "F"'] &
\end{tikzcd}$$

$F(u_1, \ldots, u_n) := [1 : u_1, \ldots, u_n].$

$F^{-1} \circ \Phi \circ id(x_0, \dots, x_n)$

$= F^{-1} \circ \Phi(x_0, \dots, x_n)$

$= F^{-1}([x_0 : \cdots : x_n])$

$= F^{-1}([1 : \frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0}])$ $(x_0 \neq 0)$.

$= \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$.

$$D(F^{-1} \circ \Phi \circ id) = \begin{bmatrix} \frac{\partial}{\partial x_0}\left(\frac{x_i}{x_0}\right) & \frac{\partial}{\partial x_j}\left(\frac{x_i}{x_0}\right) \end{bmatrix} \quad 1 \leq i, j \leq n.$$

$$= \begin{bmatrix} -\frac{x_i}{x_0^2} & \frac{\delta_{ij}}{x_0} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{x_1}{x_0^2} & \\ \vdots & \frac{1}{x_0} I \\ -\frac{x_n}{x_0^2} & \end{bmatrix}}_{n+1} \Big\} n.$$

cols are linearly indep.

col rank $= n$ $\Leftarrow$ ... max possib.

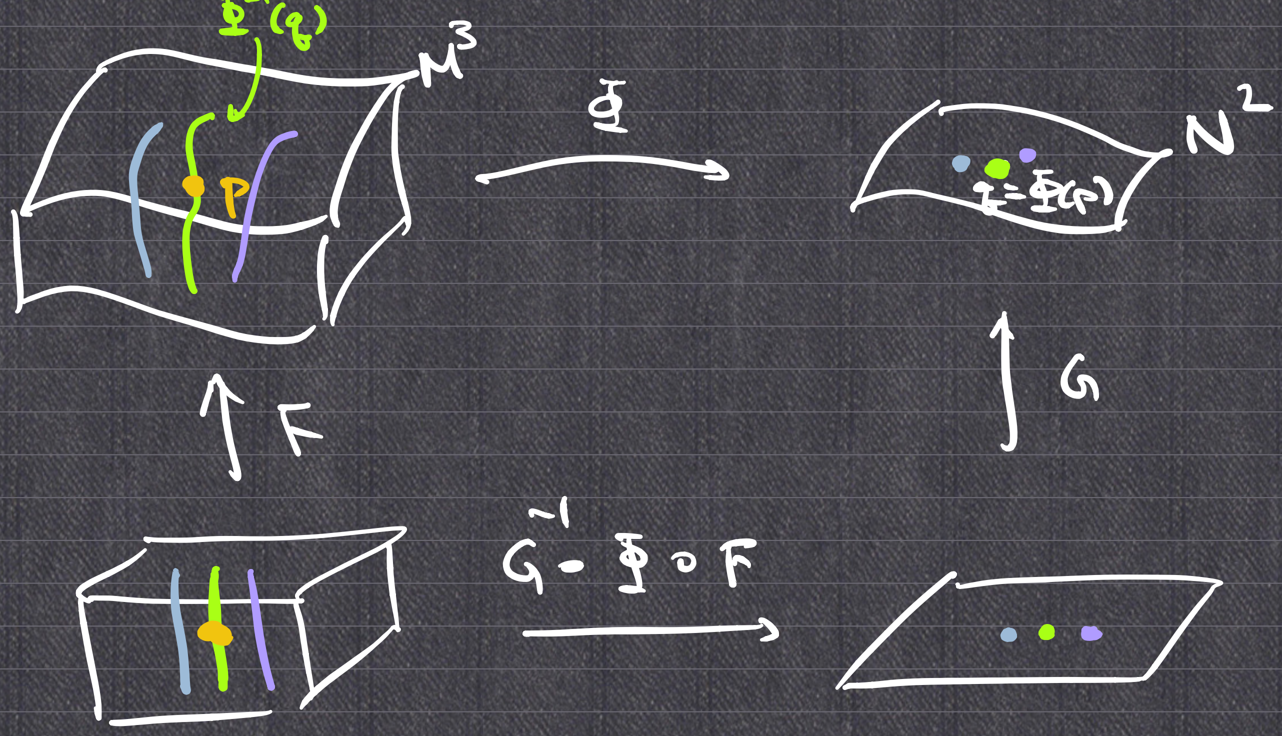
$\Rightarrow D(F^{-1} \circ \Phi \circ id)$ represents a surjective

<u>Submersion Theorem.</u>

Given $\Phi : M^{n+k} \to N^n$ is a submersion at $p$. ($p \in M^{n+k}$)

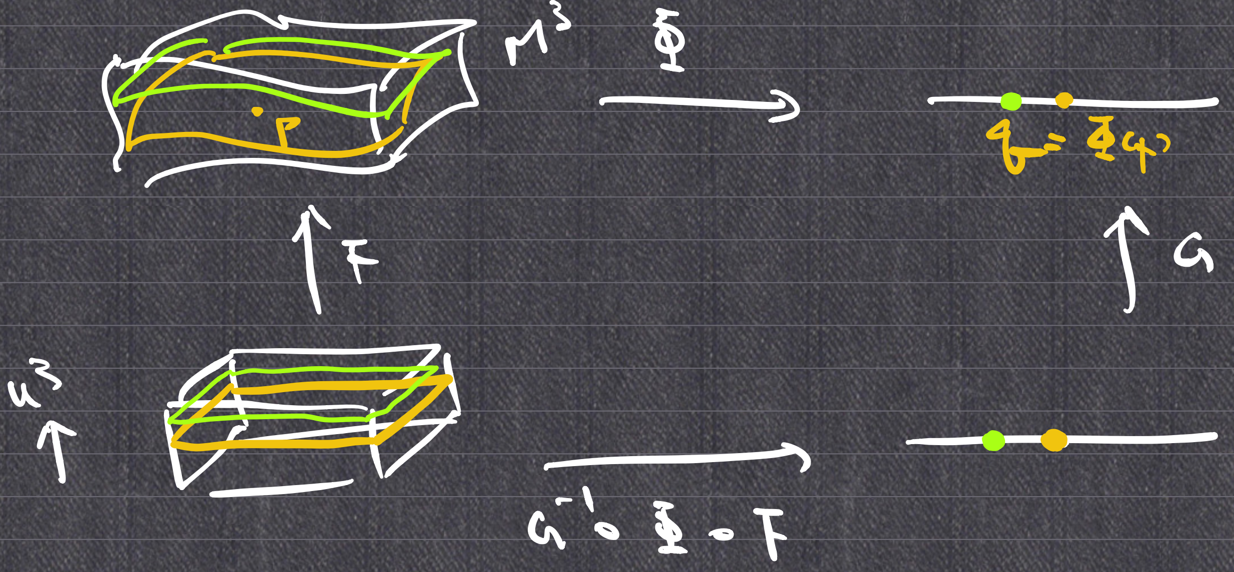
$\exists$ $F : \mathcal{U} \to M^{n+k}$ covering $p$.
$G : \mathcal{V} \to N^n$ covering $\Phi(p)$ s.t.

$$G^{-1} \circ \Phi \circ F(u_1, \dots, u_n, u_{n+1}, \dots, u_{n+k}) = (u_1, \dots, u_n).$$




$$G^{-1}\circ\Phi\circ F(u_1,u_2,u_3) = (u_1,u_2)$$

---




$$G^{-1}\circ\Phi\circ F(\underbrace{u_1,u_2}_{\longleftrightarrow},u_3) = u_3 .$$

## §2.6 - Submanifolds.

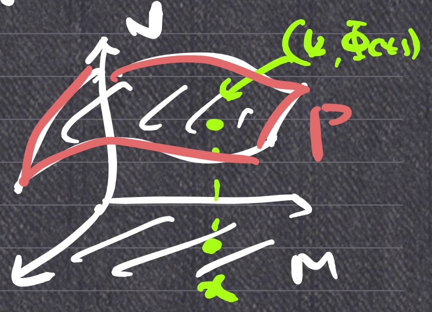
Given $M$ is a $C^\infty$ manifold, $\underset{\neq \emptyset}{N \subset M}$.

Say $N$ is a submanifold of $M$

$\overset{\text{def}}{\iff}$ $\begin{cases} N \text{ is a } C^\infty \text{ manifold} \\ \iota : N \to M \text{ is an immersion.} \\ \quad p \mapsto p \end{cases}$

e.g. $\Phi : M^m \to N^n$, $C^\infty$.

Consider $\Gamma := \{(x, \Phi(x)) : x \in M\} \subset M \times N$




$F : U \to M$

$\rightsquigarrow \tilde{F} : U \to \Gamma$

$$\tilde{F}(u_1, \dots, u_m) := \Big( \underbrace{F(u_1, \dots, u_m)}_{\in M}, \underbrace{\Phi(F(u_1, \dots, u_m))}_{\in N} \Big).$$

$G : V \to N$

$F \times G : U \times V \to M \times N$

$$(F \times G)(u_1, \dots, u_m, v_1, \dots, v_n) := \big( \underbrace{F(u_1, \dots, u_m)}_{\in M}, \underbrace{G(v_1, \dots, v_n)}_{\in N} \big)$$

Need:

$$\begin{tikzcd} \Gamma \arrow[r, "\iota"] & M \times N \\ {} \arrow[u, "\tilde{F}"] \arrow[r, "?"] & {} \arrow[u, "F \times G"'] \end{tikzcd}$$

$\iota : \Gamma \to M \times N$ is an immersion.

$(F\times G)^{-1}\circ \iota \circ \tilde{F}(u_1, \dots u_m)$

$= (F\times G)^{-1}\big( F(u_1,\dots,u_m),\ \Phi(F(u_1,\dots,u_m))\big)$

$= \big( \underbrace{u_1,\dots,u_m}_{\in\mathbb{R}^m},\ \underbrace{G^{-1}\circ\Phi\circ F(u_1,\dots,u_m)}_{\in\mathbb{R}^n}\big)$

$G\big(G^{-1}\circ\Phi\circ F(u_1,\dots,u_m)\big) = \Phi\circ F(u_1,\dots,u_m)$

$$D\big((F\times G)^{-1}\circ\iota\circ\tilde{F}\big) = \begin{array}{c} \begin{bmatrix} \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline D(G^{-1}\circ\Phi\circ F) \end{bmatrix} \\ \underbrace{\qquad\qquad\qquad}_{m} \end{array} \begin{array}{l} \}\, m \\ \\ \}\, n \\ \\ \end{array}$$

cols are linearly indep.

$D\big((F\times G)^{-1}\circ\iota\circ\tilde{F}\big)$ represents an injective linear map.

$\therefore\ \iota$ is an immersion

$\Rightarrow\ \Gamma$ is a submanifold on $M\times N$.

---

$D(G^{-1}\circ\Phi\circ F_1)$ injective

$D(G^{-1}\circ\Phi\circ F_2) = D\big(G^{-1}\circ\Phi\circ F_1\circ(F_1^{-1}\circ F_2)\big)$

$= D(G^{-1}\circ\Phi\circ F_1)\, D(F_1^{-1}\circ F_2)$

injective

invertible.

$\Rightarrow$ injective

$C =$  is a smooth manifold $\subset \mathbb{R}^3$

$z = \sqrt{x^2+y^2}$

$$\{ F(x,y) = (x,y,\sqrt{x^2+y^2}) \}$$

$id^{-1} \circ \iota \circ F(x,y) = (x,y,\sqrt{x^2+y^2})$ is not differentiable (at $(0,0)$)

$\therefore$ $\iota$ is not an immersion at $(0,0)$.

$\therefore$  is not a $C^\infty$ submanifold of $\mathbb{R}^3$.

---

$C = \{ (t, |t|) : t \in \mathbb{R} \} \subset \mathbb{R}^2$

is a smooth 1-mfd

not a smooth submfd of $(\mathbb{R}^2, \mathcal{D}_{id})$

$G(x,y) = (x, y + |x|)$.

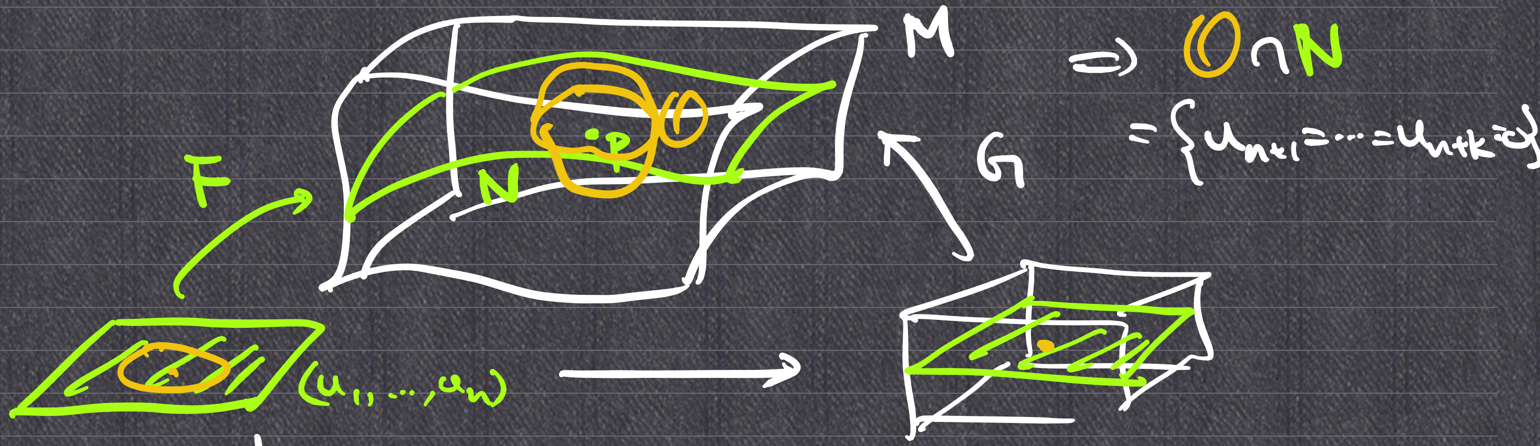
$F(t) = (t, |t|) : \mathbb{R} \to C$

$(\mathbb{R}^2, \mathcal{D}_G) \neq (\mathbb{R}^2, \mathcal{D}_{Id})$

$C$ is a smooth submfd of $(\mathbb{R}^2, \mathcal{D}_G)$.

because $G^{-1} \circ \iota \circ F(t) = G^{-1}(t, |t|) = (t, 0)$.

$$D(G^{-1} \circ \iota \circ F) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$




$$G^{-1} \circ \iota \circ F(\underline{u_1, \ldots, u_n}) = (u_1, \ldots, u_n, 0, \ldots, 0).$$

N submanifold of M

$\iota : N \to M$ is an immersion.

---

Given $\Phi : M^m \to N^n$ is a submersion on $\underbrace{\Phi^{-1}(q)}_{\neq \emptyset}$.

$\boxed{\emptyset \neq \Phi^{-1}(q) \subset M^m}$ $\Rightarrow$ $\Phi^{-1}(q)$ is a submanifold of M with $\dim \Phi^{-1}(q) = \underline{m-n}$.

$q \in N$

---

e.g. $f : \mathbb{R}^3 \to \mathbb{R}$, $\Sigma = f^{-1}(0)$ submersion on $\Sigma$

$\Leftrightarrow \nabla f(p) \neq 0 \quad \forall p \in \Sigma$.

$\Rightarrow \Sigma$ is a regular surface ($\dim = 3-1$).

$S^3 = \{(x_1, \dots, x_4) \in \mathbb{R}^4 : \ x_1^2 + \dots + x_4^2 = 1\} \subset \mathbb{R}^4.$

Let $f(x_1, \dots, x_4) : \mathbb{R}^4 \to \mathbb{R}$

$f(x_1, \dots, x_4) = x_1^2 + \dots + x_4^2 \Rightarrow S^3 = f^{-1}(1).$

$[f_*] = [2x_1 \ 2x_2 \ 2x_3 \ 2x_4] \neq [0 \ 0 \ 0 \ 0]$
on $(x_1, \dots, x_4) \in S^3.$

[ because $(0, 0, 0, 0) \notin S^3 = f^{-1}(1)$. ]

$\Rightarrow f_{*p}$ is a submersion $\forall p \in S^3$.

$\therefore f^{-1}(1)$ is smooth submfd of $\mathbb{R}^4$
and $\dim f^{-1}(1) = 4 - 1 = 3.$