MATH 4033 • Spring 2021 • Calculus on Manifolds
Problem Set #2 • Abstract Manifolds • Due Date: 21/03/2019, 11:59PM

Instruction for LaTeX-typers: OK to draw pictures by hand, and use the includegraphics command to incorporate the diagram into your homework.

1. (20 points) Consider the following equivalence relation $\sim$ defined on $\mathbb{R}^2$:

$$(x,y) \sim (x',y') \iff (x',y') = \big((-1)^n x + m,\ (-1)^m y + n\big) \text{ for some integers } m \text{ and } n.$$

   (a) Sketch an edge-identified square to represent the quotient space $\mathbb{R}^2/\sim$.

   (b) By drawing pictures, explain why $\mathbb{R}^2/\sim$ is homeomorphic to $\mathbb{RP}^2$. No need to write down the explicit homeomorphism.

   (c) Consider the two parametrizations of $\mathbb{R}^2/\sim$:

$$\begin{aligned} G_1 : (0,1)\times(0,1) &\to \mathbb{R}^2/\sim \\ (x,y) &\mapsto [(x,y)] \end{aligned} \qquad \begin{aligned} G_2 : (0,1)\times(0.5,1.5) &\to \mathbb{R}^2/\sim \\ (x,y) &\mapsto [(x,y)] \end{aligned}$$

   Find the transition map $G_2^{-1} \circ G_1$.

2. (20 points) Consider two C^∞ scalar functions $f, g : \mathbb{R}^{n\geq 3} \to \mathbb{R}$, and their non-empty level sets $\Sigma_f := f^{-1}(0)$ and $\Sigma_g := g^{-1}(0)$. Suppose $p \in \Sigma_f \cap \Sigma_g$ is a point such that $\{\nabla f(p), \nabla g(p)\}$ are linearly independent vectors in $\mathbb{R}^n$.

   (a) Show that $\Sigma_f \cap \Sigma_g$ is locally a C^∞ manifold near p, i.e. there exists an open set U in $\mathbb{R}^n$ containing p such that $\Sigma_f \cap \Sigma g \cap U$ is a C^∞ manifold. What is its dimension?

   (b) Show also that the set $\Sigma_f \cap \Sigma_g \cap U$ in (a) is a submanifold of Σ_f.

3. (20 points) The famous quintic Calabi-Yau 3-fold in string theory is the following subset in $\mathbb{CP}^4$:

$$M := \left\{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^4 : z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0\right\}.$$

   Show that M is a 6-dimensional submanifold of $\mathbb{CP}^4$ (which is 8 dimensional).

   [Hint: First be careful that $\Phi([z_0 : z_1 : z_2 : z_3 : z_4]) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 : \mathbb{CP}^4 \to \mathbb{C}$ is NOT well-defined! Try to show $M \cap F_i(\mathcal{U})$ is a submanifold of $\mathbb{CP}^4$ for each i where F_i's are the standard local coordinate charts of $\mathbb{CP}^4$.]

4. (30 points) A *Lie group* G is a smooth manifold such that multiplication and inverse maps

$$\begin{aligned} \mu : G \times G &\to G \\ (g,h) &\mapsto gh \end{aligned} \qquad \begin{aligned} \nu : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

   are both smooth (C^∞) maps. As an example, $\mathrm{GL}(n,\mathbb{R})$ is a Lie group since it is an open subset of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$, hence it can be globally parametrized using coordinates of $\mathbb{R}^{n^2}$. The multiplication map is given by products and sums of coordinates in $\mathbb{R}^{n^2}$, hence it is smooth. The inverse map is smooth too by the Cramer's rule $A^{-1} = \frac{1}{\det(A)}\text{adj(A)}$ and that $\det(A) \neq 0$ for any $A \in \mathrm{GL}(n,\mathbb{R})$.

   (a) Recall that $T_{(e,e)}(G \times G)$ can be identified with $T_eG \oplus T_eG = \{(X,Y) : X, Y \in T_eG\}$.

      i. Show that the tangent map of μ at (e,e) is given by:

$$(\mu_*)_{(e,e)}(X,Y) = X + Y.$$

      ii. Show that μ is a submersion at (e,e).

(b) Show that the tangent map of ν at e is given by:

$$(\nu_*)_e(X) = -X.$$

[Hint for part (a): when taking partial derivative $\frac{\partial f}{\partial u}$ at $(u, v) = (u_0, v_0)$, it is OK to substitute $v = v_0$ first, and then differentiate $f(u, v_0)$ by u. It is possible to prove (b) using the result from (a)i and the manifold chain rule in an appropriate way.]