MATH 4033 • Spring 2019 • Calculus on Manifolds
Problem Set #2 • Abstract Manifolds • Due Date: 10/03/2019, 11:59PM

1. (30 points) Let Σ be a regular surface in $\mathbb{R}^3$ such that $(0,0,0) \notin \Sigma$. For each $p \in \Sigma$, define $N_p\Sigma$ to be the 1-dimensional vector space spanned by a non-zero normal vector to Σ at p. Consider the set:

$$N\Sigma := \{(p, n_p) \in \{p\} \times N_p\Sigma : p \in \Sigma\},$$

and the following subset of $\Sigma \times \mathbb{R}^3$:

$$L\Sigma := \{(p, tp) \in \Sigma \times \mathbb{R}^3 : t \in \mathbb{R}\}.$$

   (a) Show that $N\Sigma$ is a C^∞ 3-manifold.

   (b) Show that $L\Sigma$ is a C^∞ 3-manifold, and is a submanifold of $\Sigma \times \mathbb{R}^3$.

   (c) Suppose further that there exists a well-defined C^∞ map $\hat{\nu} : \Sigma \to \mathbb{R}^3$, where $\hat{\nu}(p)$ is a unit normal vector to Σ at p. Show that $N\Sigma$ and $L\Sigma$ are diffeomorphic.
   [Hint: If any non-zero vectors v, w in $\mathbb{R}^3$ are parallel to each other, then $v = \frac{\langle v,w\rangle}{\|w\|^2} w$.]

2. (25 points) Consider $\Phi : \mathbb{RP}^2 \to \mathbb{R}^4$ be given by

$[\lambda x : \lambda y : \lambda z]$

$$\Phi(\underbrace{[x : y : z]}) = \frac{(x^2 - y^2, xy, zx, yz)}{x^2 + y^2 + z^2}.$$

$\Phi([x_1 : y_1 : z_1])$
$= \Phi([x_2 : y_2 : z_2])$

   (a) Show that Φ is well-defined, and is injective.

   (b) Cover $\mathbb{RP}^2$ by the standard coordinate charts. Compute the local coordinate expressions of Φ respect each coordinate charts.

   (c) Compute the matrix representation $[\Phi_*]$ with respect to **each** local coordinates chart of $\mathbb{RP}^2$.

   (d) Show that Φ is an immersion. $\Leftrightarrow$ Φ_* is injective

3. (20 points) The famous quintic Calabi-Yau 3-fold in string theory is the following subset in $\mathbb{CP}^4$:

$$M := \left\{[z_0 : z_1 : z_2 : z_3 : z_4] \in \mathbb{CP}^4 : z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0\right\}.$$

Show that M is a 6-dimensional submanifold of $\mathbb{CP}^4$ (which is 8 dimensional).

[Hint: First be careful that $\Phi([z_0 : z_1 : z_2 : z_3 : z_4]) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 : \mathbb{CP}^4 \to \mathbb{C}$ is NOT well-defined! Try to show $M \cap F_i(\mathcal{U})$ is a submanifold of $\mathbb{CP}^4$ for each i where F_i's are the standard local coordinate charts of $\mathbb{CP}^4$.]

4. (25 points) Let $p(x_1, \cdots, x_k)$ be a m-homogeneous polynomial of k variables where $k, m \geq 2$, i.e.

$$p(\lambda x_1, \cdots, \lambda x_k) = \lambda^m p(x_1, \cdots, x_k)$$

for any $\lambda > 0$, $(x_1, \cdots, x_k) \in \mathbb{R}^k$.

   (a) Prove that for any $(x_1, \cdots, x_k) \in \mathbb{R}^k$, the following identity holds:

$$\sum_{i=1}^k x_i \frac{\partial p}{\partial x_i}(x_1, \cdots, x_k) = mp(x_1, \cdots, x_k).$$

   (b) Show that for any $a \neq 0$, the level-set $p^{-1}(a)$, whenever non-empty, is a $(k-1)$-submanifold of $\mathbb{R}^k$.

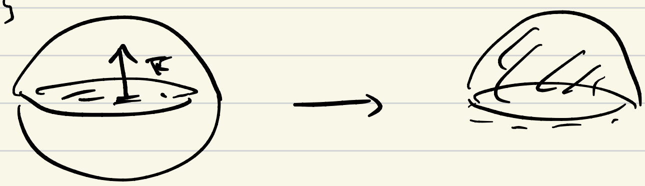
   (c) Show that for any $a \neq 0$, we have $p^{-1}(a)$ is diffeomorphic to $p^{-1}(a/|a|)$ (or both are empty).

$$\Phi : S^n \longrightarrow \mathbb{RP}^n \longleftarrow G$$

$$\Big\uparrow F(u_1,\dots,u_n) = (u_1,\dots,u_n,\sqrt{1-u_1^2-\cdots-u_n^2})$$

$(u_1,\dots,u_n)$

$\{|\vec{u}|<1\}$



$G(x_1,\dots,x_n) = [x_1:\dots,x_n:1]$

$$G^{-1}\circ\Phi\circ F(u_1,\dots,u_n)$$

$$= G^{-1}\circ\Phi\Big(u_1,\dots,u_n,\underbrace{\sqrt{1-u_1^2-\cdots-u_n^2}}_{\sqrt{1-|\vec{u}|^2}}\Big)$$

$\vec{u} = (u_1,\dots,u_n)$

$|\vec{u}| = \sqrt{u_1^2+\cdots+u_n^2}$

$$= G^{-1}\Big(\big[u_1:\dots:u_n:\sqrt{1-|\vec{u}|^2}\,\big]\Big)$$

$$= G^{-1}\left(\left[\frac{u_1}{\sqrt{1-|\vec{u}|^2}}:\dots:\frac{u_n}{\sqrt{1-|\vec{u}|^2}}:1\right]\right)$$

$$= \left(\frac{u_1}{\sqrt{1-|\vec{u}|^2}},\dots,\frac{u_n}{\sqrt{1-|\vec{u}|^2}}\right)$$

$$D(G^{-1}\circ\Phi\circ F) = \left[\frac{\partial}{\partial u_j}\frac{u_i}{\sqrt{1-|\vec{u}|^2}}\right]$$

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$$\frac{\partial}{\partial u_j}(1-|\vec{u}|^2) = \frac{\partial}{\partial u_j}(1-u_1^2-\cdots-u_n^2)$$

$$\frac{\partial}{\partial u_j}\frac{u_i}{\sqrt{1-|\vec{u}|^2}} = \frac{\sqrt{1-|\vec{u}|^2}\,\delta_{ij} - u_i\frac{\partial}{\partial u_j}\sqrt{1-|\vec{u}|^2}}{1-|\vec{u}|^2}$$

$$= \frac{\sqrt{1-|\vec{u}|^2}\,\delta_{ij} - u_i\cdot\frac{1}{2}(1-|\vec{u}|^2)^{-\frac{1}{2}}(-2u_j)}{1-|\vec{u}|^2}$$

$$= \frac{1}{\sqrt{1-|\vec{u}|^2}}\underbrace{\left(\delta_{ij}+\frac{u_iu_j}{1-|\vec{u}|^2}\right)}$$

$$I + \frac{1}{1-|\vec{u}|^2}\underbrace{\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}}_{\vec{u}}\underbrace{\begin{bmatrix}u_1 & \cdots & u_n\end{bmatrix}}_{\vec{u}^T} = I + \frac{\vec{u}\,\vec{u}^T}{1-|\vec{u}|^2}.$$

WANT: eigenvalues of $I + \frac{\vec u \vec u^T}{1-|\vec u|^2}$?

$$[u_1 \cdots u_n]\begin{bmatrix}u_1\\ \vdots \\ u_n\end{bmatrix}.$$

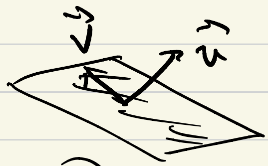
$$\left(I + \frac{\vec u\vec u^T}{1-|\vec u|^2}\right)\vec u = \vec u + \frac{\vec u(\vec u^T\vec u)}{1-|\vec u|^2}, \qquad \vec u^T\vec u = |\vec u|^2$$

$$= \left(1+\frac{|\vec u|^2}{1-|\vec u|^2}\right)\vec u = \frac{1}{1-|\vec u|^2}\,\vec u$$

$$\operatorname{Spec}\left(I + \frac{\vec u\vec u^T}{1-|\vec u|^2}\right) = \left\{ \underbrace{\frac{1}{1-|\vec u|^2}}_{+}, \ldots \right.$$

Other eigenvectors:

$$\frac{u\vec u^T}{1-|\vec u|^2}\vec v = \frac{\vec u(\overbrace{\vec u\cdot\vec v}^{0})}{1-|\vec u|^2} = 0$$



$\vec u \cdot \vec v = 0$

$\vec u^T\vec v = 0$.

$$\underbrace{0, \ldots, 0}_{n-1}$$

$\vec v_1 \quad \ldots \quad \vec v_{n-1}$

$$\left(I + \frac{\vec u\vec u^T}{1-|\vec u|^2}\right)\vec v = \vec v + 0\vec v = 1\vec v.$$

$$\operatorname{Spec}\left(I + \frac{\vec u\vec u^T}{1-|\vec u|^2}\right) = \left\{ \underbrace{\frac{1}{1-|\vec u|^2}, 1, \ldots, 1}_{\neq 0} \right\}.$$

$\therefore\ I + \frac{\vec u\vec u^T}{1-|\vec u|^2}$ is invertible.

$$\Phi([x:y:z]) := \frac{(x^2-y^2, xy, zx, yz)}{x^2+y^2+z^2}. \in \mathbb{R}^4 (w_1, w_2, w_3, w_4)$$

$$\uparrow \mathrm{id}$$

$$F(u,v) := [1:u:v]$$

$$\mathrm{id}^{-1} \circ \Phi \circ F(u,v)$$

$$= \mathrm{id}^{-1} \circ \Phi([1:u:v])$$

$$= \frac{(1-u^2, u, v, uv)}{1+u^2+v^2}$$

$$[\Phi_*] = D(\mathrm{id}^{-1} \circ \Phi \circ F)$$

$$= \begin{bmatrix} \frac{(1+u^2+v^2)(-2u)-(1-u^2)(2u)}{(1+u^2+v^2)^2} & -\frac{(1-u^2)\cdot 2v}{(1+u^2+v^2)^2} \\ \frac{(1+u^2+v^2)\cdot 1 - u\cdot 2u}{(1+u^2+v^2)^2} & -\frac{u\cdot 2v}{(1+u^2+v^2)^2} \\ -\frac{v\cdot 2u}{(1+u^2+v^2)^2} & \frac{(1+u^2+v^2)-v\cdot 2v}{(1+u^2+v^2)^2} \\ \frac{(1+u^2+v^2)\cdot v - uv\cdot 2u}{(1+u^2+v^2)^2} & \frac{(1+u^2+v^2)u - uv\cdot 2v}{(1+u^2+v^2)^2} \end{bmatrix}$$

$$= \frac{1}{(1+u^2+v^2)^2} \begin{bmatrix} 2u\overbrace{(-1\cancel{-u^2}-v^2-1\cancel{+u^2})}^{-2-v^2} & (-1+u^2)2v \\ 1-u^2+v^2 & -u\cdot 2v \\ -v\cdot 2u & 1+u^2-v^2 \\ (1-u^2+v^2)v & (1+u^2-v^2)u \end{bmatrix}$$