$$\begin{array}{l} M \ni p \\ \quad \cap \\ \quad O \\ \nearrow F(u_1,\dots,u_m) \\ \mathcal{U} \end{array}$$

$$T_pM := \operatorname{span}\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^{m}$$

$$\frac{\partial F}{\partial u_i}(p) \leftrightarrow \frac{\partial}{\partial u_i}(p)$$

$$\frac{\partial}{\partial u_i}(p) : f \mapsto \frac{\partial f}{\partial u_i}(p)$$

$$\begin{tikzcd} M^m \arrow[r, "\Phi"] & N^n \\ \mathcal{U} \arrow[u, "F(u_1,\dots,u_m)"] \arrow[r, "G^{-1}\circ\Phi\circ F"] & \mathcal{V} \arrow[u, "G(v_1,\dots,v_n)"'] \end{tikzcd}$$

$$G^{-1}\circ\Phi\circ F(u_1,\dots,u_m) = (\underbrace{v_1,\dots,v_n}_{\text{functions of }(u_1,\dots,u_m).})$$

$$\frac{\partial\Phi}{\partial u_i} = \sum_\alpha \frac{\partial v_\alpha}{\partial u_i}\frac{\partial}{\partial v_\alpha}$$

$\Phi_{*p} : T_pM \to T_{\Phi(p)}N$ linear

$$\Phi_*\left(\frac{\partial}{\partial u_i}\right) := \frac{\partial\Phi}{\partial u_i}$$

$$\Phi : \mathbb{RP}^1\times\mathbb{RP}^2 \to \mathbb{RP}^5$$

$$([x_0:x_1],[y_0:y_1:y_2]) \mapsto [x_0y_0 : x_0y_1 : x_0y_2 : x_1y_0 : x_1y_1 : x_1y_2]$$

$\nearrow F(u,v_1,v_2) := ([u:1],[v_1:1:v_2]$

$\uparrow G(w_1,\dots,w_5) = [w_1:w_2:\dots:w_5:1]$

$$\begin{aligned} &G^{-1}\circ\Phi\circ F(u,v_1,v_2) \\ &= G^{-1}\circ\Phi([u:1],[v_1:1:v_2]) \\ &= G^{-1}([uv_1 : u : uv_2 : v_1 : 1 : v_2]) \\ &= G^{-1}\left(\left[\frac{uv_1}{v_2} : \frac{u}{v_2} : u : \frac{v_1}{v_2} : \frac{1}{v_2} : 1\right]\right) \end{aligned}$$

$$= \left(\frac{uv_1}{v_2}, \frac{u}{v_2}, u, \frac{v_1}{v_2}, \frac{1}{v_2}\right)$$

$$\Phi_*\left(\frac{\partial}{\partial u}\right) = \frac{\partial \Phi}{\partial u} = \frac{v_1}{v_2}\frac{\partial}{\partial w_1} + \frac{1}{v_2}\frac{\partial}{\partial w_2} + \frac{\partial}{\partial w_3}$$

$$\Phi_*\left(\frac{\partial}{\partial v_1}\right) = \frac{\partial \Phi}{\partial v_1} = \frac{u}{v_2}\frac{\partial}{\partial w_1} + \frac{1}{v_2}\frac{\partial}{\partial w_4}$$

$$\Phi_*\left(\frac{\partial}{\partial v_2}\right) = \frac{\partial \Phi}{\partial v_2} = -\frac{uv_1}{v_2^2}\frac{\partial}{\partial w_1} - \frac{u}{v_2^2}\frac{\partial}{\partial w_2} - \frac{v_1}{v_2^2}\frac{\partial}{\partial w_4} - \frac{1}{v_2^2}\frac{\partial}{\partial w_5}.$$

$$[\Phi_*] = \begin{bmatrix} | & | & | \\ \left[\Phi_*\left(\frac{\partial}{\partial u}\right)\right] & \vdots & \vdots \\ | & | & | \end{bmatrix} = \begin{bmatrix} v_1/v_2 & u/v_2 & -uv_1/v_2^2 \\ 1/v_2 & 0 & -u/v_2^2 \\ 1 & 0 & 0 \\ 0 & 1/v_2 & -v_1/v_2^2 \\ 0 & 0 & -1/v_2^2 \end{bmatrix}$$

$$\Phi_{*_p} : T_pM \to T_{\Phi(p)}N.$$

<u>Check:</u> If $X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial u_i} = \sum_{j=1}^{m} \tilde{X}^j \frac{\partial}{\partial v_j}$

then $\sum_{i=1}^{m} X^i \frac{\partial \Phi}{\partial u_i} = \sum_{j=1}^{m} \tilde{X}^j \frac{\partial \Phi}{\partial v_j}$ $\blacksquare$

$$\begin{tikzcd} p \in M^m \arrow[r, "\Phi"] \arrow[rr, bend right, "\psi \circ \Phi"'] & N^n \arrow[r, "\psi"] & P^k \end{tikzcd}$$

$F(u_i)$ $G(v_j)$ $H(w_l)$

Φ: Phi

ψ: Psi

<u>Chain rule:</u> $(\psi \circ \Phi)_{*_p} = \psi_{*_{\Phi(p)}} \circ \Phi_{*_p}$

<u>Proof:</u> $(\psi \circ \Phi)_{*_p} : T_pM \to T_{\psi(\Phi(p))}P$

$$(\psi \circ \Phi)_*\left(\frac{\partial}{\partial u_i}\right) = \underline{\sum_j \frac{\partial w_j}{\partial u_i} \frac{\partial}{\partial w_j}}$$

$$\psi_*\left(\Phi_*\left(\frac{\partial}{\partial u_i}\right)\right) = \psi_*\left(\sum_l \frac{\partial v_l}{\partial u_i}\frac{\partial}{\partial v_l}\right)$$
$$= \sum_l \frac{\partial v_l}{\partial u_i}\frac{\partial \psi}{\partial v_l}$$
$$= \sum_l \frac{\partial v_l}{\partial u_i}\left(\sum_j \frac{\partial w_j}{\partial v_l}\frac{\partial}{\partial w_j}\right)$$
$$= \underline{\sum_j \frac{\partial w_j}{\partial u_i}\frac{\partial}{\partial w_j}}.$$

$$\therefore\ (\psi\circ\Phi)_* = \psi_*\circ\Phi_*. \qquad \blacksquare$$

Remark: $id: M \to M$

$F \nearrow \qquad \nwarrow F$

$F^{-1}\circ id\circ F = id\,(u_1,\dots,u_m) = (u_1,\dots,u_m)$

$$id_{*p}: T_pM \to T_pM \qquad \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1\end{bmatrix}.$$

Cor: If M and N are diffeomorphic, then Φ_* is invertible, and $\dim M = \dim N$.

Proof: $M \underset{\Phi^{-1}}{\overset{\Phi}{\rightleftarrows}} N$

$$\Phi\circ\Phi^{-1} = id_N$$
$$\underbrace{(\Phi\circ\Phi^{-1})_*}_{\shortparallel} = id_{TN}$$
$$\Phi_*\circ(\Phi^{-1})_* = id_{TN}.$$

Similarly:

$$\Phi^{-1}_*\circ\Phi_* = id_{TM}.$$

$\Phi_*: T_pM \to T_{\Phi(p)}N$ is invertible as a linear map.

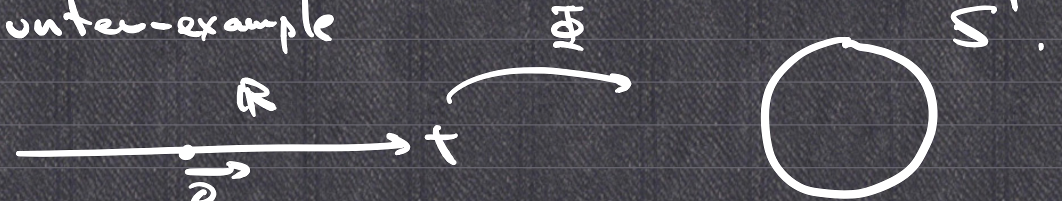
$\Rightarrow \dim T_pM = \dim T_{\Phi(p)}N$

$\Rightarrow \dim M = \dim N.$

Converse? If $\Phi_{*p} : T_pM \to T_{\Phi(p)}N$ is invertible $\forall p \in M$, then whether $\Phi : M \to N$ is invertible ?

<u>Ans:</u> <u>No!</u>

Counter-example



$\frac{\partial}{\partial t}$ $\Phi(t) = (\cos t, \sin t)$

$$\Phi_*\left(\frac{\partial}{\partial t}\right) = \frac{\partial \Phi}{\partial t} = (-\sin t, \cos t) \neq 0.$$

$\Rightarrow$ $[\Phi_*] = [\,*\,]$ $\rightsquigarrow$ Φ_{*_t} is invertible $\forall t \in \mathbb{R}$.

non-zero

But Φ is not injective.

Definition: $\Phi : M \to N$

Say Φ is a local diffeomorphism near $p \in M$

$\overset{\text{def}}{\Leftrightarrow}$ $\exists\, \mathcal{O} \ni p$ s.t. $\Phi|_{\mathcal{O}} : \mathcal{O} \to \Phi(\mathcal{O})$ is a diffeomorphism.

<u>Theorem</u> (Inverse Function Theorem, manifold version)

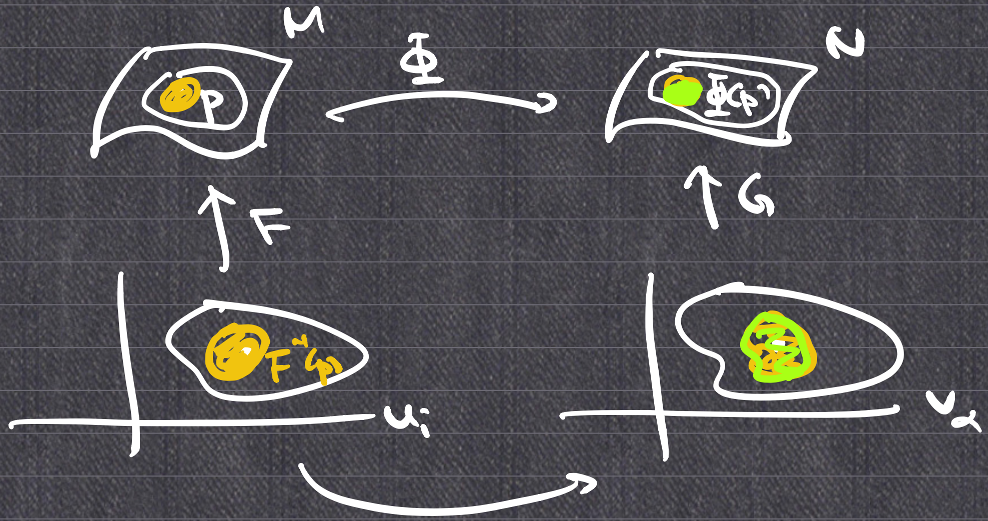
$\Phi : M \to N$ is a local diffeomorphism near $p \in M$

$\Leftrightarrow$ $\Phi_{*p} : T_pM \to T_{\Phi(p)}N$ is invertible.

<u>Proof:</u> Key observation:

$$G^{-1} \circ \Phi \circ F(u_1, \dots, u_n) = (v_1, \dots, v_n)$$

$$[\Phi_{*p}] = \frac{\partial(v_1, \dots, v_n)}{\partial(u_1, \dots, u_m)}$$



$$G^{-1}\circ\Phi\circ F(u_1,\dots,u_m)=(v_1,\dots,v_n)$$

Φ_{*_p} is invertible $\Rightarrow \det \left.\dfrac{\partial(v_1,\dots,v_n)}{\partial(u_1,\dots,u_m)}\right|_{F^{-1}(p)} \neq 0$

$\Rightarrow G^{-1}\circ\Phi\circ F$ is locally invertible near $F^{-1}(p)$.

<u>e.g.</u> $\Sigma=\{(r\cos\theta, r\sin\theta, \theta) : r>0,\ \theta\in\mathbb{R}\}$

is locally diffeomorphic to $\mathbb{R}^2\setminus\{0\}$.

<u>Claim:</u> $\pi:\Sigma\to\mathbb{R}^2\setminus\{0\}$. is a local diffeomorphism.

$(r\cos\theta, r\sin\theta,\theta)\mapsto(r\cos\theta, r\sin\theta)$

$$F(r,\theta)=(r\cos\theta, r\sin\theta, \theta) : \underset{r}{(0,\infty)}\times\underset{\theta}{\mathbb{R}}\to\Sigma.$$

$$[\pi_*]=D(\mathrm{id}^{-1}\circ\pi\circ F)$$

$$\begin{tikzcd} \Sigma \arrow[r, "\pi"] & \mathbb{R}^2\setminus\{0\} \\ (r,\theta) \arrow[u, "F"] \arrow[r, bend right, "\mathrm{id}^{-1}\circ\pi\circ F"'] & \mathbb{R}^2\setminus\{0\} \arrow[u, "\mathrm{id}"'] \end{tikzcd}$$

$\mathrm{id}^{-1}\circ\pi\circ F(r,\theta)$

$=\mathrm{id}^{-1}\circ\pi(r\cos\theta, r\sin\theta,\theta)$

$=\mathrm{id}^{-1}(r\cos\theta, r\sin\theta)$

$=(r\cos\theta, r\sin\theta)$

$$D(\mathrm{id}^{-1}\circ\pi\circ F) = \begin{bmatrix} \frac{\partial(r\cos\theta)}{\partial r} & \frac{\partial(r\cos\theta)}{\partial\theta} \\ \frac{\partial(r\sin\theta)}{\partial r} & \frac{\partial(r\sin\theta)}{\partial\theta} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \leftarrow \text{invertible}$$

$\det = r > 0.$

$\Rightarrow \pi_{*p}$ is invertible $\forall p \in \Sigma$.

$\overset{\text{IFT}}{\Rightarrow}$ $\pi : \Sigma \to \mathbb{R}^2\setminus\{0\}$ is a local diffeomorphism.

§2.5 Immersions and Submersions

<u>Definition:</u> $\Phi : M \to N$

Say Φ is an <u>immersion</u> at p

$\overset{\text{def}}{\Longleftrightarrow}$ $\Phi_{*p} : T_pM \to T_{\Phi(p)}N$ is <u>injective</u>.

$$\begin{bmatrix} 0 & A0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

RREF

$T : V \to W$ is injective

$[T] = A$ $\Longleftrightarrow$ $\ker(T) = \{0\}$.

$\Longleftrightarrow$ cols are linear independent.

$c_1\,\mathrm{col}_1 + c_2\,\mathrm{col}_2 + \cdots + c_n\,\mathrm{col}_n = 0.$

$$\begin{bmatrix} | & | & & | \\ \mathrm{col}_1 & \mathrm{col}_2 & \cdots & \mathrm{col}_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$$

<u>e.g.</u> $M^2 \subset \mathbb{R}^3$ regular surface in $\mathbb{R}^3$.

$\iota : M^2 \to \mathbb{R}^3$ (inclusion map)

$F \nearrow p \mapsto p$ $\uparrow \mathrm{id}$

(u,v) (x,y,z)

$F(u,v) = (x(u,v), y(u,v), z(u,v))$

$$\mathrm{id}^{-1} \circ \iota \circ F(u,v) = \mathrm{id}^{-1} \circ \iota\big(x(u,v), y(u,v), z(u,v)\big)$$
$$= \big(x(u,v), y(u,v), z(u,v)\big)$$

$$[\iota_*] = D\left(\mathrm{id}^{-1} \circ \iota \circ F\right) = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \leftarrow \text{represents an injective linear map}$$

$$\Big\{ \frac{\partial F}{\partial u}, \frac{\partial F}{\partial v} \Big\} \text{ linearly indep.}$$

ι_* is injective.

$\therefore$ ι is an immersion at every $p \in M^2$.

$\uparrow$

\iota