$$
\begin{tikzcd}
M \ni p \in O \arrow[r, "f"] & \mathbb{R} \\
\mathcal{U} \ni F^{-1}(p) \arrow[u, "F"] \arrow[ur, "f\circ F"'] &
\end{tikzcd}
$$

f is C^k at p $\overset{\text{def}}{\Longleftrightarrow}$ $f\circ F$ is C^k at $F^{-1}(p)$ for some local parametrization $F(u_i)$ covering p.

$$\frac{\partial f}{\partial u_i}(p) := \frac{\partial}{\partial u_i}(f\circ F)\Big|_{F^{-1}(p)}$$

$G(v_\alpha)$

$$
\begin{aligned}
\frac{\partial f}{\partial v_\alpha}(p) &:= \frac{\partial}{\partial v_\alpha}(f\circ G)\Big|_{G^{-1}(p)} \\
&= \frac{\partial}{\partial v_\alpha}(f\circ F)\circ(F^{-1}\circ G) \\
&= \sum_i \frac{\partial (f\circ F)}{\partial u_i}\frac{\partial u_i}{\partial v_\alpha} \\
&= \sum_i \frac{\partial u_i}{\partial v_\alpha}\frac{\partial f}{\partial u_i}(p)
\end{aligned}
$$

$$
\begin{array}{c}
f \\
| \\
u_i \\
| \\
v_\alpha
\end{array}
$$

$$
\begin{tikzcd}
M \ni p \arrow[r, "\Phi"] & N \ni \Phi(p) \\
\mathcal{U} \arrow[u, "F(u_i)"] \arrow[r, "G^{-1}\circ\Phi\circ F"'] & \mathcal{V} \arrow[u, "G"']
\end{tikzcd}
$$

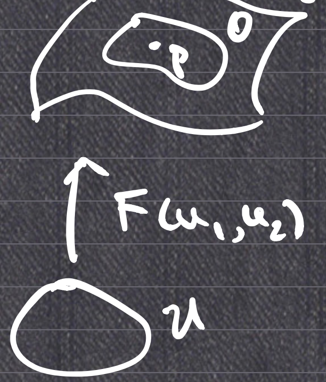
Regular surface: $\dfrac{\partial \Phi}{\partial u_i} = \dfrac{\partial}{\partial u_i}(\Phi\circ F)$

$\Phi\circ F : \underset{\mathbb{R}^2}{\overset{\cap}{\mathcal{U}}} \to N \subset \mathbb{R}^3$

$\Phi\circ F = (x(u,v), y(u,v) \ \ldots$

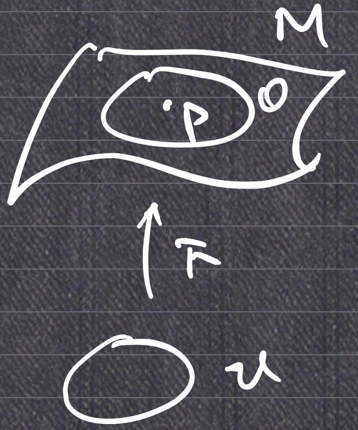
- Tangent vectors / tangent space of an abstract manifold?

Regular surface $\subset \mathbb{R}^3$



$F(u_1, u_2) = (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$

$$T_pM = \operatorname{span}\left\{\frac{\partial F}{\partial u_i}(p)\right\}_{i=1}^{2}$$



abstract manifold

tangent vectors:

$$\frac{\partial}{\partial u_i}(p)$$

$$f \mapsto \frac{\partial f}{\partial u_i}(p)$$

$$\frac{\partial}{\partial u_i}(p) : \underbrace{C^1(O, \mathbb{R})}_{\substack{\text{set of } C^1 \\ \text{functions} \\ \text{from } O \text{ to } \mathbb{R}}} \to \mathbb{R}$$

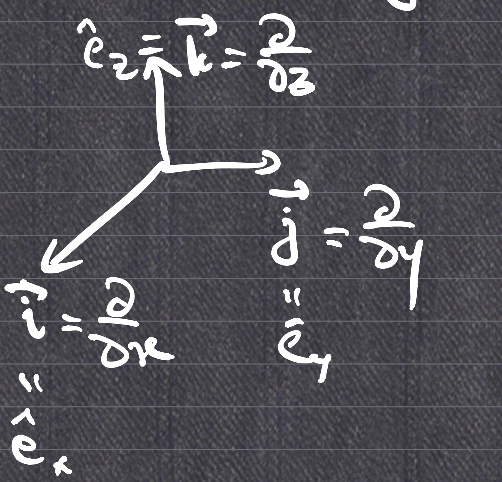
$$\underbrace{T_pM}_{\text{tangent space of } M \text{ at } p} := \operatorname{span}\left\{\frac{\partial}{\partial u_i}(p)\right\}.$$

tangent space of M at p.

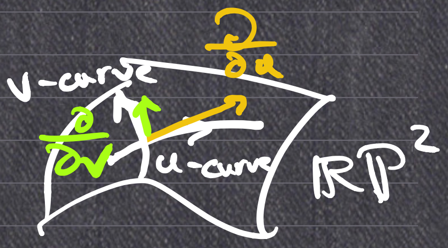
$$\frac{\partial F}{\partial u_i}(p) \longleftrightarrow \frac{\partial}{\partial u_i}(p)$$

Regular surfaces | Manifold

$\mathbb{R}^3$ (x, y, z)



$$\begin{array}{c} \mathbb{RP}^2 \\ \uparrow F(u,v) := [1:u:v] \\ \mathbb{R}^2 \end{array}$$



$F : \mathcal{U} \to \mathcal{O} \subset M$

$\;\;\;\; = F(u_1, \ldots, u_n)$

$$T_pM = \operatorname{span}\left\{ \frac{\partial}{\partial u_i}(p) \right\}_{i=1}^{n}$$

$$\operatorname{span}\left\{ \frac{\partial}{\partial v_\alpha}(p) \right\}_{\alpha=1}^{n} \overset{?}{=} \operatorname{span}\left\{ \frac{\partial}{\partial u_i}(p) \right\}_{i=1}^{n}$$

$$\frac{\partial f}{\partial v_\alpha} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial v_\alpha} \frac{\partial f}{\partial u_i} \qquad \forall f \in C^1(\underset{p}{\overset{\cup}{\mathcal{O}}}, \mathbb{R}).$$

$$\Rightarrow \frac{\partial}{\partial v_\alpha} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial v_\alpha} \frac{\partial}{\partial u_i}$$

$$\Rightarrow \operatorname{span}\left\{ \frac{\partial}{\partial v_\alpha}(p) \right\}_{\alpha=1}^{n} < \operatorname{span}\left\{ \frac{\partial}{\partial u_i}(p) \right\}_{i=1}^{n}$$

Similarly, $\operatorname{span}\left\{ \frac{\partial}{\partial u_i}(p) \right\}_{i=1}^{n} < \operatorname{span}\left\{ \frac{\partial}{\partial v_\alpha}(p) \right\}_{\alpha=1}^{n}$

Proposition: $\left\{\frac{\partial}{\partial u_i}(p)\right\}_{i=1}^{n}$ are linearly independent.

Proof: Assume $\exists\, c_1, \dots, c_n \in \mathbb{R}$

s.t. $c_1 \frac{\partial}{\partial u_1}(p) + \dots + c_n \frac{\partial}{\partial u_n}(p) = 0.$

$u_i : O \to \mathbb{R}$

$$\Rightarrow \quad c_1 \underbrace{\frac{\partial u_i}{\partial u_1}(p)}_{0} + \dots + c_n \underbrace{\frac{\partial u_i}{\partial u_n}(p)}_{0} = 0$$

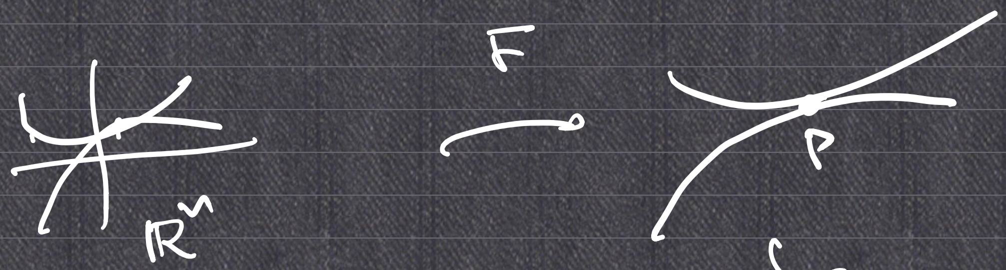
$\frac{\partial u_i}{\partial u_i} = 1$

$c_i \frac{\partial u_i}{\partial u_i} = 0 \Rightarrow c_i = 0$

$\forall i \in \{1, 2, \dots, n\}.$ ∎

Cor:

$\dim M = \dim T_pM$



$$T_pM := \left\{ \begin{array}{l} \text{space of } C^1 \text{ curves} \\ \text{through } p \end{array} \right\} \Big/ \sim$$

- Partial derivative of $\Phi : M \to N$ smooth mfds.

$$\begin{array}{ccc} & \Phi : M \to N & \\ F(u_i) \nearrow & & \uparrow G(v_\alpha) \end{array}$$

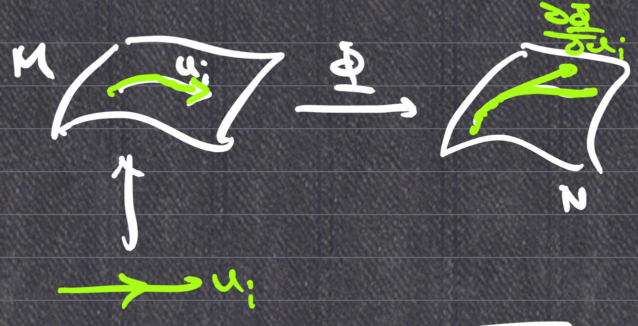
$\frac{\partial \Phi}{\partial u_i} \in T_{\Phi(p)}N$

Regular surfaces:

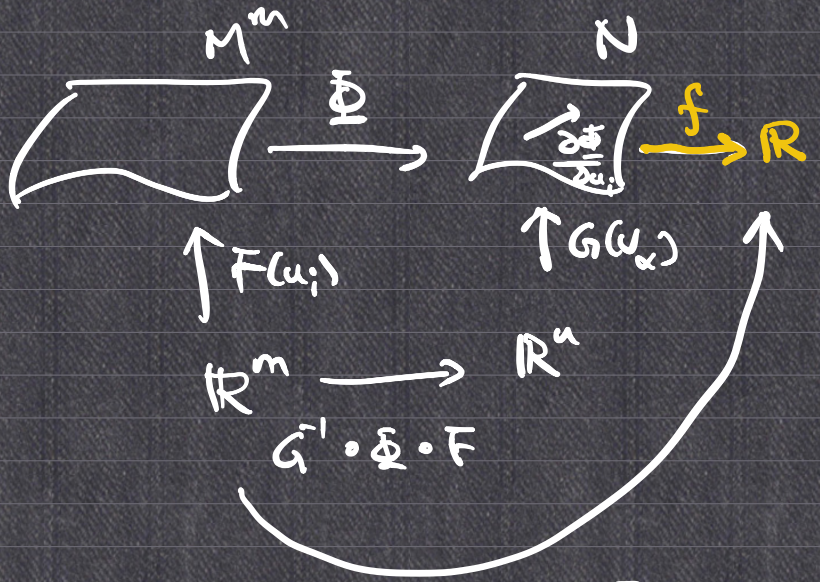
$\frac{\partial \Phi}{\partial u_i} := \frac{\partial}{\partial u_i}(\Phi \circ F)$

$\in T_{\Phi(p)}N$

$$\frac{\partial \Phi}{\partial u_i} : C^1(N, \mathbb{R}) \to \mathbb{R}.$$
$$f \mapsto ?$$



$$\boxed{\frac{\partial \Phi}{\partial u_i}\cdot f := \frac{\partial}{\partial u_i}(f\circ \Phi \circ F)}$$



$$f\circ\Phi\circ F$$

$$\frac{\partial \Phi}{\partial u_i}\cdot f$$
$$= \frac{\partial}{\partial u_i}(f\circ\Phi\circ F)$$
$$= \frac{\partial}{\partial u_i}(f\circ G)\circ(G^{-1}\circ\Phi\circ F)$$
$$= \sum_{\alpha=1}^{n} \frac{\partial (f\circ G)}{\partial v_\alpha}\frac{\partial v_\alpha}{\partial u_i}$$
$$= \sum_{\alpha=1}^{n} \frac{\partial v_\alpha}{\partial u_i}\frac{\partial f}{\partial v_\alpha} \qquad \forall C^1\ f$$

$$G^{-1}\circ\Phi\circ F(u_1,\dots,u_n)$$
$$= (v_1(u_1,\dots,u_n),\dots,v_n(u_1,\dots,u_n))$$

$$\Rightarrow \boxed{\frac{\partial \Phi}{\partial u_i} = \sum_{\alpha=1}^{n}\frac{\partial v_\alpha}{\partial u_i}\frac{\partial}{\partial v_\alpha}}$$

$$\begin{array}{l} f \\ \big| \ f\circ G \\ v_\alpha \\ \big| \ G^{-1}\circ\Phi\circ F \\ u_i \end{array}$$

e.g.

$$\Phi : \mathbb{RP}^1 \times \mathbb{RP}^2 \to \mathbb{RP}^5$$
$$\Phi([x_0:x_1],[y_0:y_1:y_2])$$
$$:= [x_0y_0 : x_0y_1 : x_0y_2 : x_1y_0 : x_1y_1 : x_1y_2]$$

$$\begin{tikzcd} & \mathbb{RP}^1\times\mathbb{RP}^2 \arrow[r, "\Phi"] & \mathbb{RP}^5 \\ (F\times G)(u_1,v_1,v_2) := ([1:u_1],[1:v_1:v_2]) \arrow[ur] & & H(w_1,\dots,w_5) = [1:w_1:\dots:w_5]. \arrow[u, "H"'] \end{tikzcd}$$

$$\frac{\partial \Phi}{\partial u_1} \overset{?}{=}$$

$$H^{-1} \circ \Phi \circ (F \times G)(u_1, v_1, v_2)$$

$$= H^{-1} \circ \Phi([1 : u_1], [1 : v_1 : v_2])$$

$$= H^{-1}([1 : \underbrace{v_1 : v_2 : u_1 : u_1 v_1 : u_1 v_2}_{}])$$

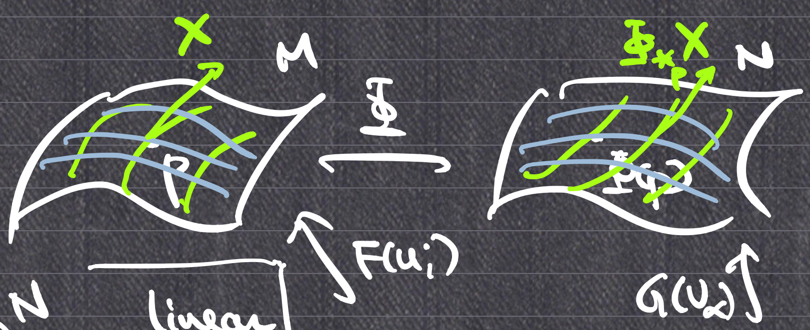
$$= (v_1, v_2, u_1, u_1 v_1, u_1 v_2) \qquad \xrightarrow{\frac{\partial}{\partial u_1}} (0, 0, 1, v_1, v_2)$$

$$\begin{array}{ccccc} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ w_1 & w_2 & w_3 & w_4 & w_5 \end{array}.$$

$$\frac{\partial \Phi}{\partial u_1} = \cancel{0 \frac{\partial}{\partial w_1}} + \cancel{0 \frac{\partial}{\partial w_2}} + \cancel{1 \cdot} \frac{\partial}{\partial w_3} + v_1 \frac{\partial}{\partial w_4} + v_2 \frac{\partial}{\partial w_5}$$

$$\frac{\partial \Phi}{\partial v_2} = 0 \frac{\partial}{\partial w_1} + 1 \cdot \frac{\partial}{\partial w_2} + 0 \frac{\partial}{\partial w_3} + 0 \frac{\partial}{\partial w_4} + u_1 \frac{\partial}{\partial w_5}.$$

$$\begin{array}{ccc} \Phi : M & \to & N \\ \Psi & & \Psi \\ p & & \Phi(p) \end{array}$$



$$\Phi_{*p} : T_p M \to T_{\Phi(p)} N$$ linear

$$\Phi_{*p}\left(\frac{\partial}{\partial u_i}(p)\right) := \frac{\partial \Phi}{\partial u_i}(p)$$

$d\Phi_p$, $T_p \Phi$.

tangent map of Φ at p.

$$[\Phi_{*p}] = \frac{\partial(v_1, \ldots, v_n)}{\partial(u_1, \ldots, u_m)}(p)$$

$$\Phi_{*p}\Big(\underbrace{\frac{\partial}{\partial u_i}}_{}(p)\Big)$$

$$= \sum_{\alpha=1}^{n} \frac{\partial v_\alpha}{\partial u_i} \underbrace{\frac{\partial}{\partial v_\alpha}}_{}(\Phi(p))$$