MATH 4033 • Spring 2017 • Calculus on Manifolds
Problem Set #2 • Abstract Manifolds • Due Date: 08/03/2017, 6PM

Instructions: *When proving a certain set is a manifold, you don't have to verify topological conditions such as Hausdorff-ness, second countability, continuity of local parametrizations. Furthermore, to reduce your workload, when proving a certain map $\Phi : M \to N$ is smooth, just verify smoothness of $G^{-1} \circ \Phi \circ F$ for one F and one G: be faithful the other combinations can be verified similarly.*

1. (15 points) The complex projective plane $\mathbb{CP}^1$ is defined as follows: $\mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$.

$$\mathbb{CP}^1 := \{[z_0 : z_1] : (z_0, z_1) \neq (0,0)\}.$$

Here z_0, z_1 are complex numbers, and we declare $[z_0 : z_1] = [w_0 : w_1]$ if and only if $(z_0, z_1) = \lambda(w_0, w_1)$ for some $\lambda \in \mathbb{C}\backslash\{0\}$.

   (a) Show that $\mathbb{CP}^1$ is a smooth manifold of (real) dimension 2.

   (b) Show that $\mathbb{CP}^1$ and the sphere $\mathbb{S}^2$ are diffeomorphic. [Hint: consider stereographic projections]

2. (20 points) Consider the following equivalence relation $\sim$ defined on $\mathbb{R}^2$:

$$(x, y) \sim (x', y') \iff (x', y') = \big((-1)^n x + m,\ y + n\big) \text{ for some integers } m \text{ and } n.$$

   (a) Sketch an edge-identified square to represent the quotient space $\mathbb{R}^2/\sim$.

   (b) Consider the two parametrizations of $\mathbb{R}^2/\sim$:

$$G_1 : (0,1) \times (0,1) \to \mathbb{R}^2/\sim \qquad\qquad G_2 : (0,1) \times (0.5, 1.5) \to \mathbb{R}^2/\sim$$
$$(x, y) \mapsto [(x, y)] \qquad\qquad (x, y) \mapsto [(x, y)]$$

   Find the transition map $G_2^{-1} \circ G_1$.

   (c) Write down a diffeomorphism between $\mathbb{R}^2/\sim$ and the Klein bottle K in $\mathbb{R}^4$ described in Example 2.16.

3. (20 points) Consider the following subset of $\mathbb{R}^2 \times \mathbb{RP}^1$

$$M = \left\{ \Big((x_1, x_2), [y_1 : y_2] \Big) \in \mathbb{R}^2 \times \mathbb{RP}^1 \,\middle|\, x_1 y_2 = y_1 x_2 \right\}$$

   (a) Show that M is a smooth 2-manifold by considering the following parametrizations:

$$F(u_1, u_2) = \Big((u_1 u_2,\ u_2), [u_1 : 1] \Big)$$
$$G(v_1, v_2) = \Big((v_1,\ v_1 v_2), [1 : v_2] \Big)$$

   (b) Consider the two projection maps $\pi_1 : M \to \mathbb{R}^2$ and $\pi_2 : M \to \mathbb{RP}^1$ defined by:

$$\pi_1\Big((x_1, x_2), [y_1 : y_2] \Big) = (x_1, x_2)$$
$$\pi_2\Big((x_1, x_2), [y_1 : y_2] \Big) = [y_1 : y_2]$$

      i. Show that $\pi_1^{-1}(p)$ is either a point, or diffeomorphic to $\mathbb{RP}^1$.
      ii. Show that π_2 is a submersion.

4. (25 points) The tangent bundle TM of a smooth n-manifold M is the disjoint union of all tangent spaces of M, i.e.

$$TM := \bigcup_{p \in M} \{p\} \times T_pM = \{(p, V_p) : p \in M \text{ and } V_p \in T_pM\}.$$

   (a) Show that TM is a smooth $2n$-manifold. [Again, skip the topological parts, but show detail work of the differentiable parts.]

   (b) Show that the map $\pi : TM \to M$ defined by $\pi(p, V_p) := p$ is a submersion.

   (c) Define the subset $\Sigma_0 := \{(p, 0_p) \in TM : p \in M\}$ where 0_p is the zero vector in T_pM. This set Σ_0 is called the zero section of the tangent bundle. Show that Σ_0 is a smooth n-manifold diffeomorphic to M, and that it is a submanifold of TM.

   (d) Now suppose M is just a C^k-manifold (where $k \geq 2$), then TM is a $C^{\text{what?}}$-manifold?

5. (20 points) A *Lie group* G is a smooth manifold such that multiplication and inverse maps

$$\begin{aligned} \mu : G \times G &\to G \\ (g, h) &\mapsto gh \end{aligned} \qquad\qquad \begin{aligned} \nu : G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are both smooth (C^∞) maps. As an example, $\mathrm{GL}(n, \mathbb{R})$ is a Lie group since it is an open subset of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$, hence it can be globally parametrized using coordinates of $\mathbb{R}^{n^2}$. The multiplication map is given by products and sums of coordinates in $\mathbb{R}^{n^2}$, hence it is smooth. The inverse map is smooth too by the Cramer's rule $A^{-1} = \frac{1}{\det(A)}\mathrm{adj(A)}$ and that $\det(A) \neq 0$ for any $A \in \mathrm{GL}(n, \mathbb{R})$.

   (a) Recall that $T_{(e,e)}(G \times G)$ can be identified with $T_eG \oplus T_eG = \{(X, Y) : X, Y \in T_eG\}$.

      i. Show that the tangent map of μ at (e, e) is given by:

$$(\mu_*)_{(e,e)}(X, Y) = X + Y.$$

      ii. Show that μ is a submersion at (e, e).

   (b) Show that the tangent map of ν at e is given by:

$$(\nu_*)_e(X) = -X.$$

[Hint for part (a): when taking partial derivative $\frac{\partial f}{\partial u}$ at $(u, v) = (u_0, v_0)$, it is OK to substitute $v = v_0$ first, and then differentiate $f(u, v_0)$ by u. It is possible to prove (b) using the result from (a)i and the manifold chain rule in an appropriate way.]

1 (a) Show $\mathbb{CP}^1$ is a C^∞ 2-manifold.

$$\mathbb{CP}^1 = \left\{ [z_0 : z_1] : (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\} \right\}.$$

$F_0(z) : \mathbb{C} \to \mathbb{CP}^1$, $\boxed{F_0(z) := [1 : z]}$

$F_1(w) : \mathbb{C} \to \mathbb{CP}^1$, $\boxed{F_1(w) := [w : 1]}$

$F_0^{-1} \circ F_1 : \mathbb{C}\setminus\{0\} \to \mathbb{C}$

$F_0(\mathbb{C}) \cap F_1(\mathbb{C}) \subset \mathbb{CP}^1$

$= \left\{ [z_0 : z_1] : z_0 \neq 0 \text{ and } z_1 \neq 0 \right\}.$

$F_0^{-1} \circ F_1(z)$

$= F_0^{-1}([z : 1])$

$= \frac{1}{z}.$

$F_0(\underset{\mathbb{C}}{?}) = [z : 1]$

$F_0(1/z) = [1 : 1/z]$

$= [z : 1].$

$F_0^{-1} \circ F_1 : \underbrace{z \mapsto \frac{1}{z}}$ is C^∞ on $\mathbb{C}\setminus\{0\}$.

(b) In lecture: $S^2 \cong \mathbb{C} \cup \{\infty\}$.

$$\mathbb{CP}^1 = \{[1 : w] : w \in \mathbb{C}\} \sqcup \{[0 : 1]\}$$

$\Psi : \mathbb{CP}^1 \to \mathbb{C} \cup \{\infty\}.$

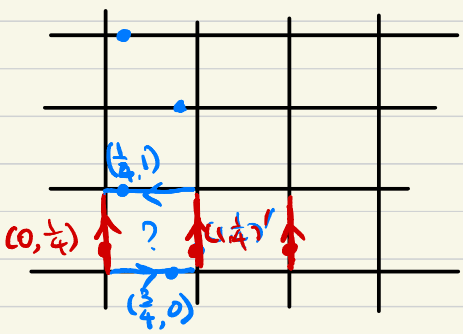
$\Psi([1 : w]) = w \in \mathbb{C}$

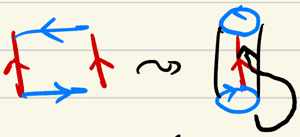
$\Psi([0 : 1]) = \infty \in \mathbb{C} \cup \{\infty\}.$

Check $? \circ \Psi^{-1} \circ ?$ C^∞.

2(a):

$(x, y) \sim (x', y') \iff (x', y') = ((-1)^n x + m,\ y + n)$ for some integers m and n.



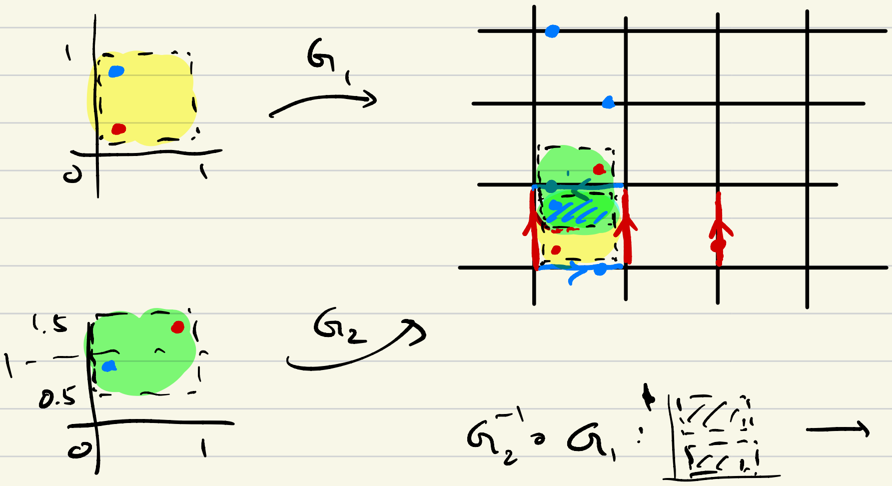


$(1, \frac{1}{4}) = ((-1)^0 0 + 1,\ \frac{1}{4} + 0)$

$(2, \frac{1}{4}) = ((-1)^0 1 + \underset{m}{1},\ \frac{1}{4} + \underset{n}{0})$

$(\frac{3}{4}, 0) = (\underbrace{(-1)^{-1}\frac{1}{4}}_{-\frac{1}{4}} + 1,\ 1 + \underset{n}{(-1)})$

(b) $G_1 : (0,1)\times(0,1) \longrightarrow \mathbb{R}^2/\sim$, $\quad (x,y) \mapsto [(x,y)]$ | $G_2 : (0,1)\times(0.5,1.5) \longrightarrow \mathbb{R}^2/\sim$, $\quad (x,y) \mapsto [(x,y)]$.



$G_2^{-1}\circ G_1 :$ $\longrightarrow \mathbb{R}^2$.

$$G_2^{-1}\circ G_1(x,y) = \begin{cases} (1-x,\ y+1) & \text{if } \\ (x,y) & \text{if } \end{cases}$$

is C^∞ on

3

$$M = \left\{ \left((x_1, x_2), [y_1 : y_2] \right) \in \mathbb{R}^2 \times \mathbb{RP}^1 \;\middle|\; x_1 y_2 = y_1 x_2 \right\}$$

Given:

$((x_1,x_2),[y_1:y_2]) \in M$

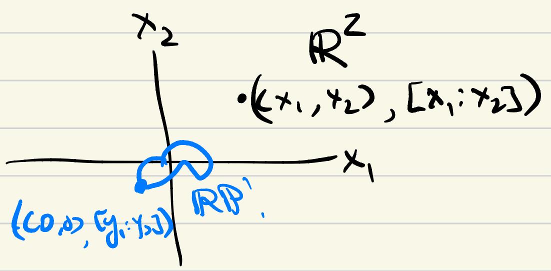
if $(x_1,x_2) \neq (0,0)$, $\quad x_1 \neq 0 \Rightarrow \dfrac{y_2}{y_1} = \dfrac{x_2}{x_1}$ $\quad y_1 \neq 0$

$\quad x_2 \neq 0 \Rightarrow \dfrac{x_1}{x_2} = \dfrac{y_1}{y_2}$ $\quad y_2 \neq 0$

$\Rightarrow [y_1 : y_2] = [x_1 : x_2]$.

if $(x_1,x_2) = (0,0)$, $\rightarrow x_1 y_2 = y_1 x_2 = 0$ always hold.

$\Rightarrow [y_1 : y_2]$ can be any point in $\mathbb{RP}^1$.



$M = \mathbb{R}^2$ blow-up at $(0,0)$.

$$M = \left\{ ((x_1,x_2),[x_1:x_2]) \mid (x_1,x_2) \neq (0,0) \right\} \longleftrightarrow \mathbb{R}^2 \setminus \{0\}.$$

$$\sqcup \left\{ ((0,0),[y_1:y_2]) \mid [y_1:y_2] \in \mathbb{RP}^1 \right\} \longleftrightarrow \mathbb{RP}^1.$$