If $\exists \{ F_\alpha : \mathcal{U}_\alpha \overset{\subset \mathbb{R}^n}{\rightarrow} \mathcal{O}_\alpha^{\subset M} \}$ homeomorphism.

① $M = \bigcup_\alpha F_\alpha(\mathcal{U}_\alpha) = \bigcup_\alpha \mathcal{O}_\alpha$

② $F_\alpha^{-1} \circ F_\beta$ is C^∞ on its domain $\forall \alpha, \beta$.

then M is called a smooth manifold.

$\mathcal{C}$

$z = \sqrt{x^2+y^2}$

$\{ F(x,y) := (x, y, \sqrt{x^2+y^2}) : \mathbb{R}^2 \to \mathcal{C} \}$

① $F(\mathbb{R}^2) = \mathcal{C}$ ✓

② $F^{-1} \circ F = id$ is C^∞ ✓.

$\mathcal{C}$ is a smooth manifold. ✓

Differential structure.

$\mathbb{R}^n \qquad \{ id : \mathbb{R}^n \to \mathbb{R}^n \}$

$\{ F(x) : \mathbb{R}^n \to \mathbb{R}^n ,\ F(x) = |x|^{\frac{1}{2}} x \}$.

but $id^{-1} \circ F(x) = |x|^{\frac{1}{2}} x$ is not C^∞.

$(\mathbb{R}^n, \{ id : \mathbb{R}^n \to \mathbb{R}^n \})$, $(\mathbb{R}^n, \{ F(x) = |x|^{\frac{1}{2}} x \})$.

$\left(S^2, \left(\{ \underbrace{F_+, F_-}_{\text{stereographic}}, F_1(x,y) = (x, y, \sqrt{1-x^2-y^2}), \text{etc.} \ldots, F_{sph.}(\theta, \varphi), \ldots \} \right) \right)$

differential structure.

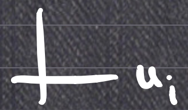
$\left(\mathbb{R}^2, \{id, F(r,\theta) = (r\cos\theta, r\sin\theta), \ldots\}\right)$

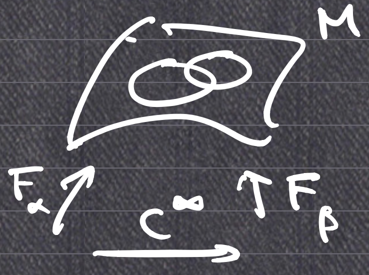
$\left(\mathbb{R}^2, \{G(x) = |x|^{\frac{1}{2}}x, \ldots\}\right)$

§2.2

$f : M \to \mathbb{R}$

$\uparrow F(u_1, \ldots, u_n)$

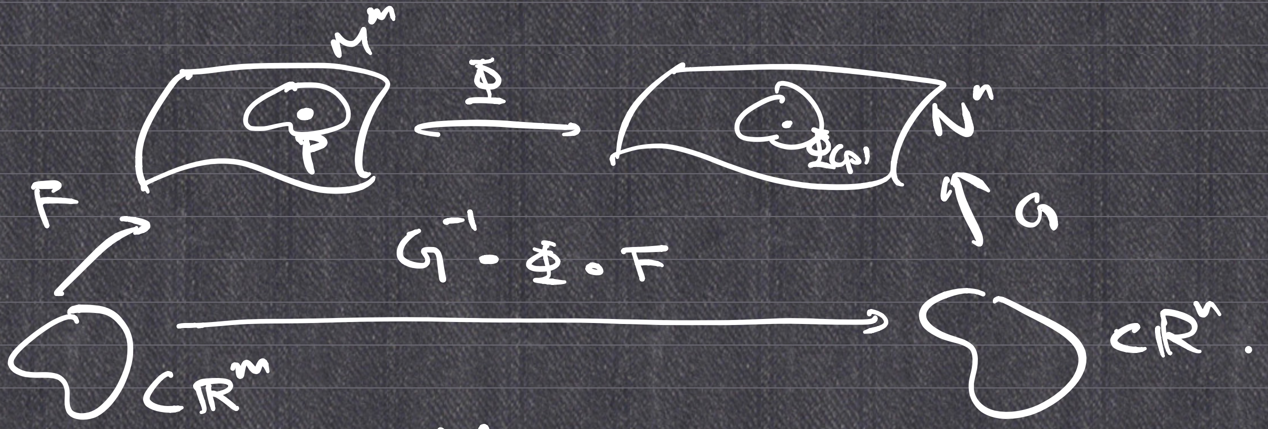




$$\underbrace{\frac{\partial f}{\partial u_i}}_{4033} := \underbrace{\frac{\partial}{\partial u_i}}_{2023}(f\circ F)$$

$$\begin{tikzcd} M \arrow[r, "f"] & \mathbb{R} \\ \mathcal{U} \subset \mathbb{R}^n \arrow[u, "F"] \arrow[ur, "f\circ F"'] & \end{tikzcd}$$

If $f\circ F$ is C^k on $\mathcal{U}$ ($\subset \mathbb{R}^n$), then say f is C^k on $F(\mathcal{U}) \subset M$.



Φ is C^k at $p \overset{def}{\Longleftrightarrow} G^{-1}\circ\Phi\circ F$ is C^k at $F^{-1}(p)$

where G covers $\Phi(p)$ and F covers p.

$$\tilde{G}^{-1}\circ\Phi\circ\tilde{F} = \underbrace{\tilde{G}^{-1}\circ G}_{C^\infty}\circ\underbrace{G^{-1}\circ\Phi\circ F}_{C^k}\circ\underbrace{F^{-1}\circ\tilde{F}}_{C^\infty}.$$

e.g.

$$\begin{tikzcd} M\times N \arrow[r, "\pi"] & M \\ (p,q) \arrow[r, mapsto] & p \\ \arrow[u, "F\times G"] \arrow[r] & \arrow[u, "F"'] \end{tikzcd}$$

$(F\times G)(\vec{u},\vec{v}) := (F(\vec{u}), G(\vec{v}))$

$$\underbrace{F^{-1}\circ \pi \circ (F\times G)}(u,v) = F^{-1}\circ \pi\,(F(u), G(v))$$
$$= F^{-1}(F(u))$$
$$= u.$$

$(u, v) \mapsto u$ is C^∞.

$\therefore$ π is C^∞.

e.g.

$f : \mathbb{R}^n \to \mathbb{R}$ is continuous.

$\Gamma_f := \{(x, f(x)) : x \in \mathbb{R}^n\}$ is C^∞ manifold.

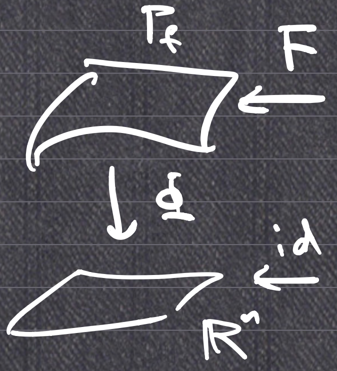
$F(x) = (x, f(x)) \in \Gamma_f : \mathbb{R}^n \to \Gamma_f$.

Claim: $\Phi(x, f(x)) := x$

$\Phi : \Gamma_f \to \mathbb{R}^n$ is C^∞

and $\Phi^{-1} : \mathbb{R}^n \to \Gamma_f$ is C^∞.

Proof:



$$\mathrm{id}^{-1}\circ \Phi \circ F(x)$$
$$= \mathrm{id}^{-1}\circ \Phi\,(x, f(x))$$
$$= \mathrm{id}^{-1}(x)$$
$$= x \quad \text{is } C^\infty.$$

$\Phi^{-1}(x) = ?$

$\Phi(?) = x$

$$F^{-1}\circ \Phi^{-1}\circ \mathrm{id}(x) = F^{-1}\circ \Phi^{-1}(x)$$
$$= F^{-1}(x, f(x)) = x \quad \text{is } C^\infty.$$

Diffeomorphism

$\Phi: M \to N$ is a diffeomorphism

$\overset{\text{def}}{\Longleftrightarrow}$ ① Φ is bijective,

② Φ is C^∞ on M,

③ $\Phi^{-1}: N \to M$ is C^∞ on N.

If $\exists\, \Phi: M \to N$ diffeomorphism between M and N,

then M and N are diffeomorphic.

e.g. $M = \mathbb{C} \cup \{\infty\}$. $(x, y) = x+yi$

$G_+(x+yi) = x+yi$, $\mathbb{R}^2 \to \mathbb{C} \subset M$

$G_-(x+yi) = \begin{cases} \frac{1}{x+yi} & \text{if } (x,y) \neq (0,0) \\ \infty & \text{if } (x,y) = (0,0). \end{cases}$

$\mathbb{R}^2 \to (\mathbb{C} \setminus \{0\}) \cup \{\infty\}$

$F_+ \nearrow S^2 \nwarrow F_-$

stereographic projections

$\Phi: M \to S^2$

$$\underset{M}{\Phi(\hat{p})} := \begin{cases} F_+(x+yi) & \text{if } p \in \mathbb{C}, \ \boxed{x+yi} \\ (0,0,1) & \text{if } p = \infty. \end{cases}$$

$F_+^{-1} \circ F_-(z) = \frac{1}{z}$

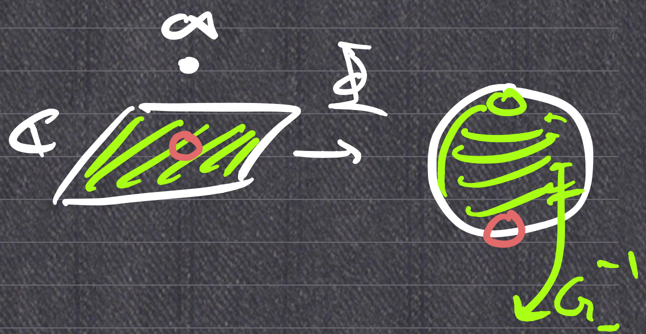
$F_-^{-1} \circ F_+(z) = \frac{1}{z}$.

<u>Claim:</u> Φ is a diffeomorphism.

<u>Proof:</u>

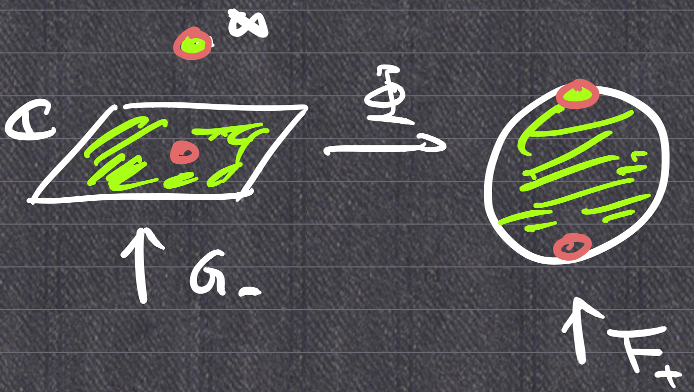
$$M \xrightarrow{\Phi} S^2$$
$$G_+ \nearrow \ \nwarrow G_- \qquad F_+ \nearrow \ \nwarrow F_-$$

$F_-^{-1} \circ \Phi \circ G_+ : \mathbb{C} \setminus \{0\} \to \cdots$



$$F_-^{-1} \circ \Phi \circ G_+ (x+yi)$$
$$= F_-^{-1} \circ \Phi (x+yi)$$
$$= F_-^{-1} \circ F_+ (x+yi)$$
$$= \frac{1}{x+yi} \text{ is } C^\infty \text{ on } \mathbb{C} \setminus \{0\}.$$

$F_+^{-1} \circ \Phi \circ G_- : \mathbb{C} \setminus \{0\} \to \cdots$



$$F_+^{-1} \circ \Phi \circ G_- (x+yi)$$
$$= F_+^{-1} \circ \Phi \left(\frac{1}{x+yi}\right)$$
$$= F_+^{-1}\left(F_+\left(\frac{1}{x+yi}\right)\right)$$
$$= \frac{1}{x+yi} \text{ is } C^\infty \text{ on } \mathbb{C} \setminus \{0\}.$$

Exercise: $F_-^{-1} \circ \Phi \circ G_-$

$F_+^{-1} \circ \Phi \circ G_+$.

$\therefore \mathbb{C} \cup \{\infty\}$ is diffeomorphic to S^2.

$\mathbb{CP}^n$ complex projective space.

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$$

$(z_0, \dots, z_n) \sim (w_0, \dots, w_n) \overset{\text{def}}{\Longleftrightarrow} \exists \lambda \in \mathbb{C} \setminus \{0\}$ s.t. $z_i = \lambda w_i \ \forall i$.

$$\mathbb{CP}^n = \left\{ [z_0 : \cdots : z_n] \,\middle|\, (z_0, \dots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\} \right\}.$$

$$\mathbb{CP}^1 = \left\{ [z_0 : z_1] \,\middle|\, (z_0, z_1) \in \mathbb{C}^2 \setminus \{0\} \right\}.$$

$[z_0 : z_1] \in \mathbb{CP}^1$
- $z_0 \neq 0 \to [z_0 : z_1] = [1 : \frac{z_1}{z_0}]$.
- $z_0 = 0 \to [0 : \underset{\neq 0}{z_1}] = [\frac{0}{z_1} : 1] = [0:1]$

$$\mathbb{CP}^1 = \underbrace{\{[1:w] : w \in \mathbb{C}\}}_{z_0 \neq 0} \cup \underbrace{\{[0:1]\}}_{z_0 = 0}$$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

$w \in \mathbb{C}$. $\qquad\qquad$ ∞.

$$\mathbb{CP}^1 \xrightarrow{\Phi} \underbrace{\mathbb{C} \cup \{\infty\}}_{} \cong S^2.$$

$$\begin{cases} [1:w] \mapsto w \\ [0:1] \mapsto \infty \end{cases}.$$

HW1 Q4: To show it is possible to parametrize S^2 s.t. transition maps are holomorphic on their domains.

$\longrightarrow$ S^2 is a complex manifold.

(or S^2 has a complex structure.)

$\cancel{S^4}$, $\cancel{S^8}$, $\cancel{S^{10}}$, $\cancel{\ldots}$

__BIG__ __OPEN__ __PROBLEM__:

Is S^6 a complex manifold?