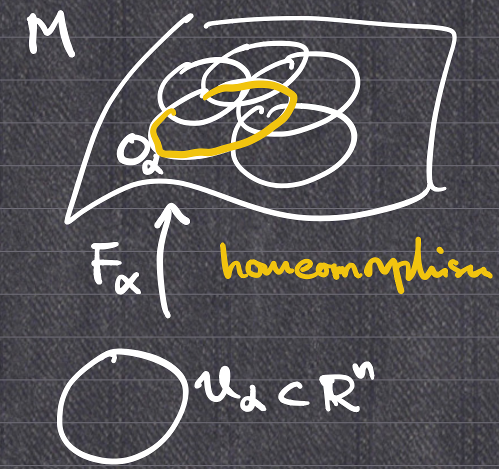
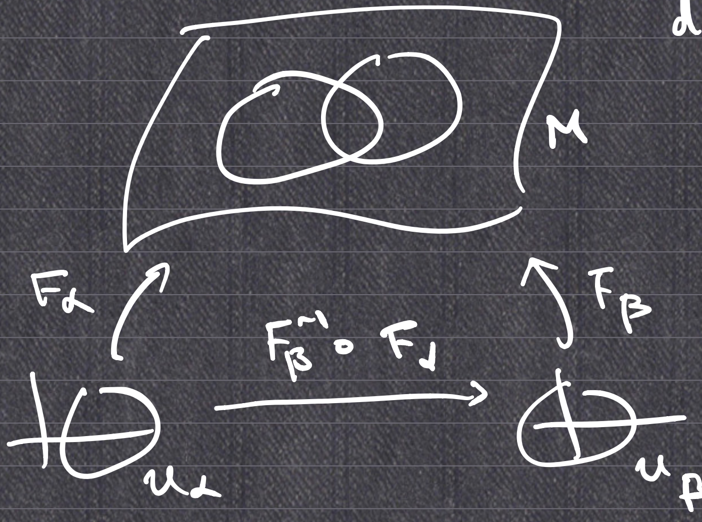


topological space

Hausdorff, second countable.

① $M = \bigcup_\alpha O_\alpha$

② $F_\beta^{-1} \circ F_\alpha$ is C^∞ on its domain $\forall \alpha, \beta$.



then call:

M is C^∞-manifold.

① $\mathbb{R}^n$ ✓

② regular surfaces

③ $M = \mathbb{C} \cup \{\infty\}$.

④ M, N smooth manifolds

$\Rightarrow M \times N := \{(x, y) : x \in M, y \in N\}$.

is a C^∞ manifold.

$$\begin{array}{ccc} O_M \subset M & \quad & O_N \subset N \\ F_\alpha \uparrow & & G_\beta \uparrow \\ U_M \subset \mathbb{R}^m & & U_N \subset \mathbb{R}^n \end{array}$$

$\dim M = m$
$\dim N = n$.

$$\{F_\alpha \times G_\beta : \underbrace{U_M \times U_N}_{\substack{U_M \quad U_N \\ \subset \mathbb{R}^{m+n}}} \to O_M \times O_N \subset M \times N\}.$$

$$(F_\alpha \times G_\beta)(\underset{\in U_M}{x}, \underset{\in U_N}{y}) := (F_\alpha(x), G_\beta(y))$$

$$(F_\gamma \times G_\eta)^{-1} \circ (F_\alpha \times G_\beta)(x,y)$$

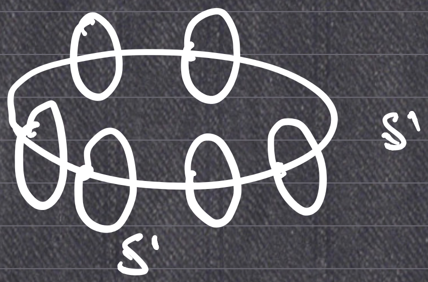
$$= (F_\gamma \times G_\eta)^{-1} (F_\alpha(x), G_\beta(y))$$

$$= (\underbrace{F_\gamma^{-1} \circ F_\alpha(x), \; G_\eta^{-1} \circ G_\beta(y)}_{C^\infty}) \qquad (F_\gamma \times G_\eta)(\,?\,) = (F_\alpha(x), G_\beta(y))$$

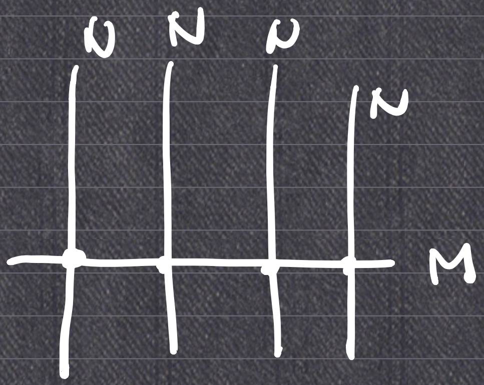
C^∞.

$\Rightarrow (F_\gamma \times G_\eta)^{-1} \circ (F_\alpha \times G_\beta)$ is C^∞

$S^1 \times S^1$



$M \times N$



$(\mathbb{R}, \sim)$

↑

equivalent relation : $x \sim y \overset{def}{\iff} x - y \in \mathbb{Z}$.

$1 \sim 2, \; 2 \sim 3$

$1 \nsim 1.5. \quad \pi \nsim e.$

$[x] := \{ y \in \mathbb{R} : y \sim x \}$

$[1] = \{ \ldots, -2, -1, 0, 1, 2, 3, \ldots \}$

$[2] = \{ \ldots, -2, -1, 0, 1, 2, 3, \ldots \}$

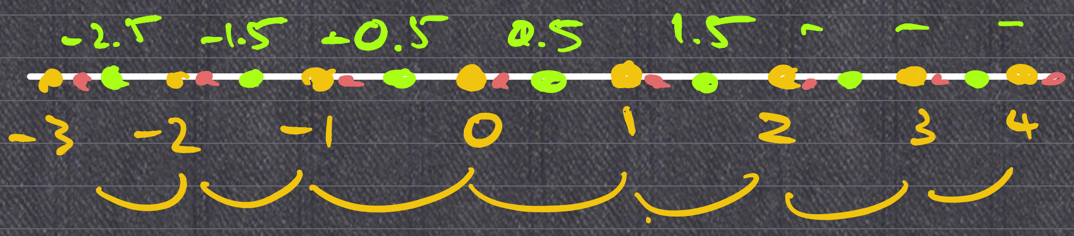
$[1]=[2]$.

$[2]+[2]=[4]$

$[2]+[2]=[5]$.

$[1.5]=\{\dots,-0.5,0.5,1.5,2.5,3.5,\dots\}\neq[1]$.

$[2.5]=[1.5]$.

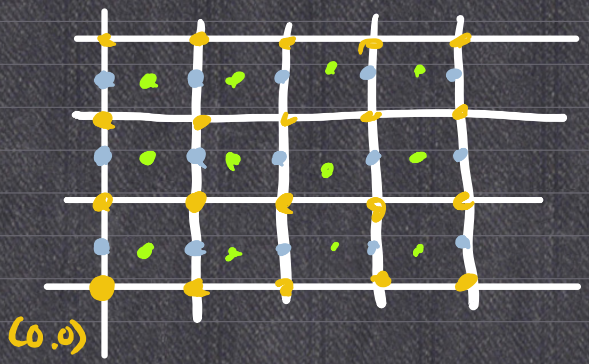


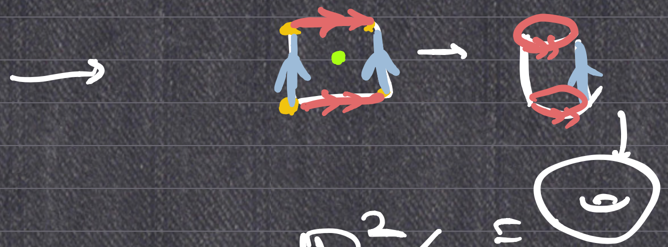
$\mathbb{R}/\sim$

$=\{[x]: x\in\mathbb{R}\}$

$\mathbb{R}^2$.

$\vec{x}\sim\vec{y} \iff \vec{x}-\vec{y}\in\mathbb{Z}^2$





$\mathbb{R}^2/\sim =$.

$[(0,0)]=\mathbb{Z}^2$.

$\mathbb{RP}^2$ = real projective plane

$=(\mathbb{R}^3-\{\vec{0}\})/\sim$

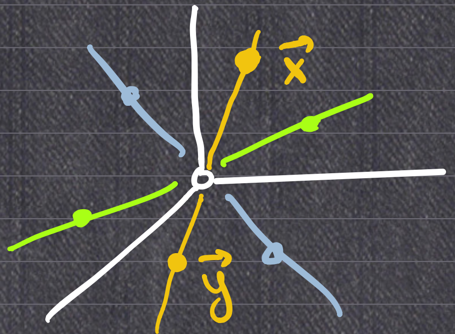
$\vec{x}\sim\vec{y} \overset{\text{def}}{\iff} \exists\lambda\neq 0$ s.t. $\vec{x}=\lambda\vec{y}$.

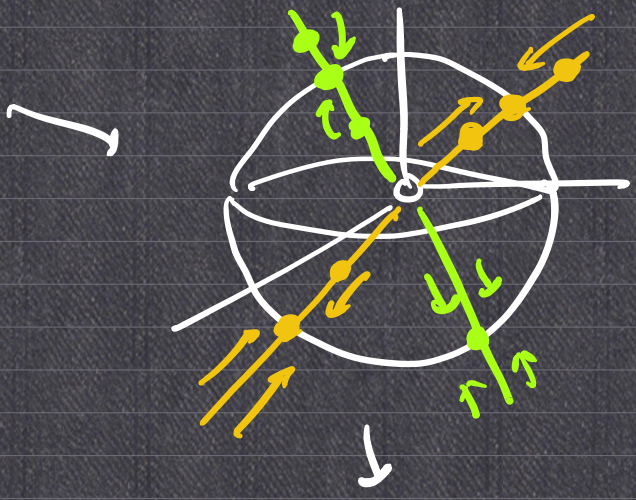
$\quad\;\;\text{"}\qquad\;\;\text{"}$

$(x_0,x_1,x_2)\;(y_0,y_1,y_2)\in\mathbb{R}^3\setminus\{\vec{0}\}$.

$(x_0, x_1, x_2) = \lambda(y_0, y_1, y_2)$.

$\mathbb{R}^3$





$\mathbb{RP}^2 = \left(\mathbb{R}^3 - \{\vec{0}\}\right)/\sim \; = \left\{ [(x_0, x_1, x_2)] : (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{\vec{0}\} \right\}$

$[(1,2,3)] = [(2,4,6)]$

$[x_0 : x_1 : x_2]$ $\qquad [1:2:3] = [2:4:6]$.

$\cancel{F(x_0, x_1, x_2) = [x_0 : x_1 : x_2]}$.

$F(1,2,3) = [1:2:3]$

$F(2,4,6) = [2:4:6] = [1:2:3]$

$[x_0 : \underset{\neq 0}{x_1} : x_2] = \left[\frac{x_0}{x_1} : 1 : \frac{x_2}{x_1}\right]$ $\qquad$ $[\underset{\neq 0}{x_0} : x_1 : x_2] = \left[1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}\right]$.

$\mathbb{RP}^2 = \mathcal{O}_0 \cup \mathcal{O}_1 \cup \mathcal{O}_2$

$\mathcal{O}_i = \{ [x_0 : x_1 : x_2] : x_i \neq 0 \}$

$F_i : \mathbb{R}^2 \to O_i \subset \mathbb{RP}^2 \quad (i=0,1,2).$

$F_0(x_1,x_2) := [1 : x_1 : x_2]$

$F_1(x_0,x_2) := [x_0 : 1 : x_2]$

$F_2(x_0,x_1) := [x_0 : x_1 : 1].$

injectivity:

$F_0(x_1,x_2) = F_0(y_1,y_2)$

$\Rightarrow [1 : x_1 : x_2] = [1 : y_1 : y_2]$

$\Rightarrow \exists \lambda \in \mathbb{R}\setminus\{0\}$ s.t. $(1, x_1, x_2) = \lambda(1, y_1, y_2)$

$\Rightarrow \lambda = 1.$

$\Rightarrow (x_1,x_2) = (y_1,y_2).$

$F_1^{-1} \circ F_0 : F_0^{-1}(O_0 \cap O_1) \to \mathbb{R}^2$

$F_1^{-1} \circ F_0(x_1,x_2)$

$= F_1^{-1}(\underbrace{[\overset{\neq 0}{1} : \overset{\neq 0}{x_1} : x_2]}_{\in O_0 \cap O_1}) \leftarrow \boxed{x_1 \neq 0.}$

$= F_1^{-1}([\frac{1}{x_1} : 1 : \frac{x_2}{x_1}])$

$= (\frac{1}{x_1}, \frac{x_2}{x_1})$

$F_1^{-1} \circ F_0$ is C^∞ on $F_0^{-1}(O_0 \cap O_1) \leftarrow \boxed{x_1 \neq 0}$

$\left.\begin{array}{l} F_2^{-1} \circ F_0 \\ F_2^{-1} \circ F_1 \\ F_0^{-1} \circ F_1 \\ F_1^{-1} \circ F_2 \\ F_0^{-1} \circ F_2 \end{array}\right\}$ similar.

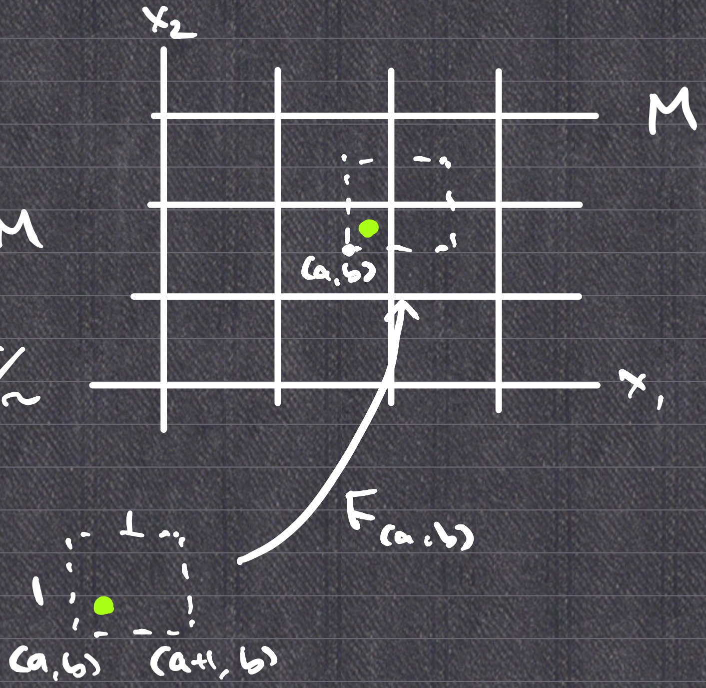
$M = \mathbb{R}^2/\sim$ $\qquad \vec{x} \sim \vec{y} \overset{\text{def}}{\iff} \vec{x}-\vec{y} \in \mathbb{Z}^2$.

$\forall (a,b)\in\mathbb{R}^2$

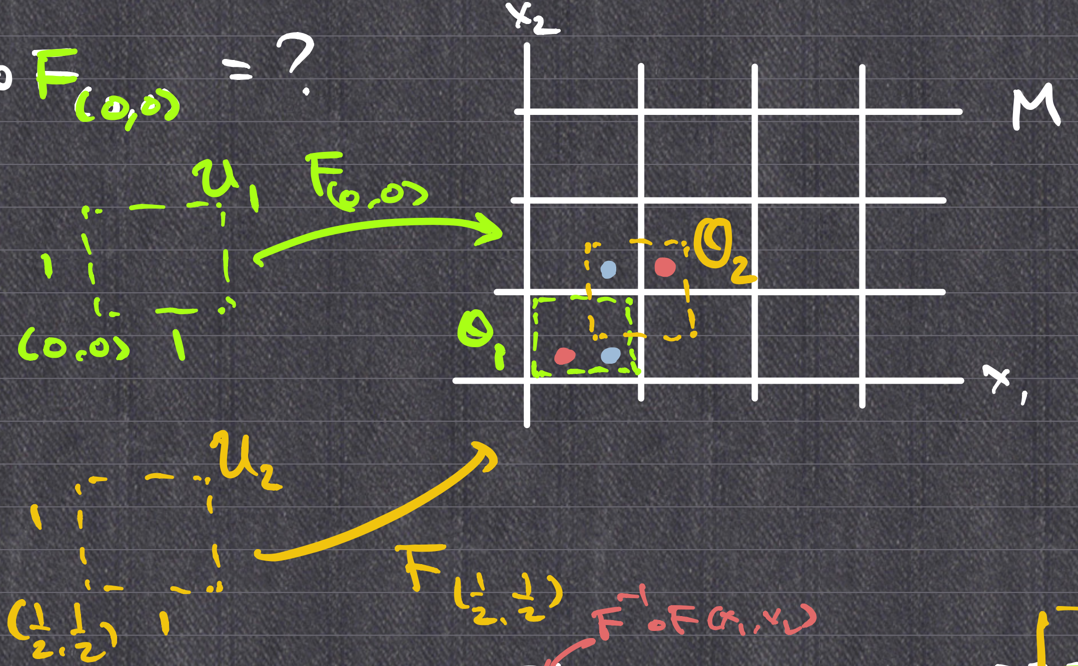
$F_{(a,b)}(x_1,x_2) : \underbrace{(a,a+1)\times(b,b+1)}_{\subset\mathbb{R}^2} \to M$

$F_{(a,b)}(x_1,x_2) := [(x_1,x_2)] \in \mathbb{R}^2/\sim$

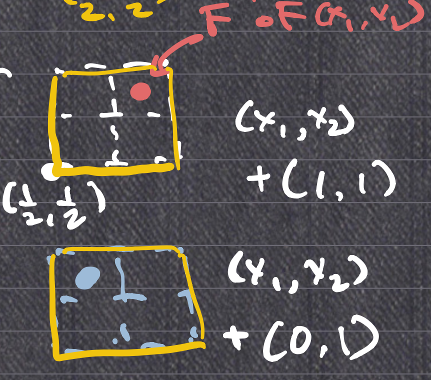
$\{F_{(a,b)}\}_{(a,b)\in\mathbb{R}^2}$.

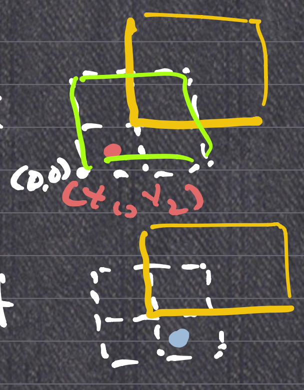


$F^{-1}_{(\frac{1}{2},\frac{1}{2})}\circ F_{(0,0)} = ?$



$F^{-1}_{(\frac{1}{2},\frac{1}{2})}\circ F_{(0,0)}(x_1,x_2) = \begin{cases} (x_1,x_2)+(1,1) & \text{if} \\ (x_1,x_2)+(0,1) & \text{if} \\ & \text{if} \\ & \text{if} \end{cases}$





is C^∞ on $F^{-1}_{(0,0)}(O_1\cap O_2)$