MATH 4033 • Spring 2021 • Calculus on Manifolds • Tutorial #1

1. Let $\mathbb{S}^2$ be the unit sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Suppose $f : \mathbb{S}^2 \to (0, \infty)$ is a smooth, positive-valued function. Consider the set Σ defined by:

$$\Sigma := \{f(x)\, x : x \in \mathbb{S}^2\}.$$

   (a) Suppose $F(u, v) : \mathcal{U} \to \mathbb{S}^2$ is a smooth local parametrization of $\mathbb{S}^2$. Show that:

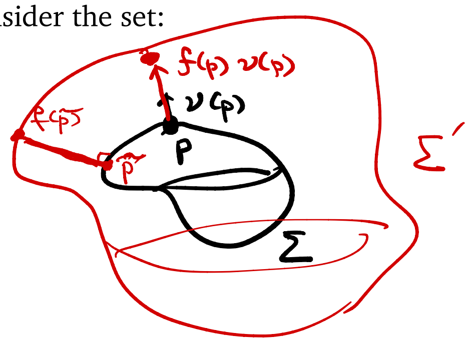
$$\begin{aligned} &G : \mathcal{U} \to \Sigma \\ &(u, v) \mapsto f\big(F(u, v)\big)\, F(u, v) \end{aligned}$$

   is a smooth local parametrization of Σ. Hence, show that Σ is a regular surface. [We now call this G the *parametrization of* Σ *induced by* F.] Let $F_i(u, v) : \mathcal{U}_i \to \mathbb{S}^2$, where $i = 1, 2$, be two overlapping smooth local parametrizations of $\mathbb{S}^2$, and $G_i : \mathcal{U}_i \to \Sigma$ be the parametrization of Σ induced by F_i. Show that $G_1^{-1} \circ G_2 = F_1^{-1} \circ F_2$.

   (b) Show that $\mathbb{S}^2$ and Σ are diffeomorphic. Write down the diffeomorphism explicitly.

2. Consider a regular surface Σ in $\mathbb{R}^3$. Denote $\nu : \Sigma \to \mathbb{R}^3$ be a smooth map of unit normal vectors to Σ. Let $f : \Sigma \to (0, \infty)$ be a C^∞ function on Σ, and consider the set:

$$\hat{\Sigma} := \{p + f(p)\nu(p) : p \in \Sigma\}.$$



   (a) Suppose $F(u_1, u_2)$ is a C^∞ local parametrization of Σ.

      i. Show that $\dfrac{\partial \nu}{\partial u_i}(p) \in T_p\Sigma$.

      ii. Consider the linear map $h : T_p\Sigma \to T_p\Sigma$ defined by:

$$h\left(\frac{\partial F}{\partial u_i}\right) := \frac{\partial \nu}{\partial u_i}$$

      and extends linearly to all of $T_p\Sigma$. Show that h is self-adjoint with respect to the standard dot product, i.e.

$$\langle h(X), Y\rangle = \langle X, h(Y)\rangle \;\text{ for any } X, Y \in T_p\Sigma.$$

   (b) From (a), we denote $h\big(\frac{\partial F}{\partial u_i}\big) = \sum_j h_i^j \frac{\partial F}{\partial u_j}$. Consider the map $\hat{F}$ defined on the same domain as F:

$$\hat{F}(u_1, u_2) := F(u_1, u_2) + f(F(u_1, u_2))\nu(F(u_1, u_2)).$$

   Suppose $\hat{F}$ is a homeomorphism onto its image. Show that if the linear map:

$$\mathrm{id} + fh : T_p\Sigma \to T_p\Sigma$$

   is invertible for any $p \in \Sigma$, then $\hat{F}$ is a C^∞ local parametrization of $\hat{\Sigma}$.

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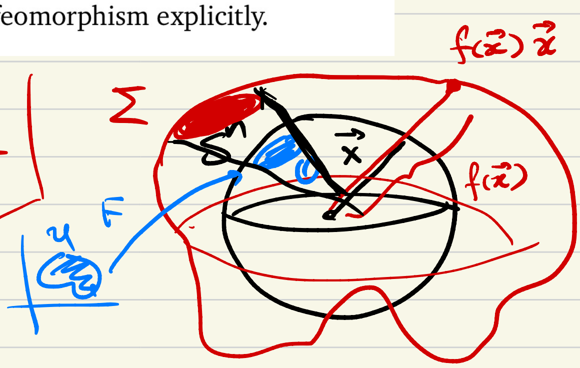
   (b) Show that $\mathbb{S}^2$ and Σ are diffeomorphic. Write down the diffeomorphism explicitly.

$F(u,v) : \mathcal{U} \to \mathcal{O}$

$G(u,v) = f\big(F(u,v)\big)\, \underbrace{F(u,v)}_{\in S^2} : \mathcal{U} \to \Sigma$



① G is C^∞ ✓

② G is injective:

If $G(u,v) = G(\tilde{u},\tilde{v})$, $\quad f(F(u,v))\, \underbrace{F(u,v)}_{\in S^2} = f(F(\tilde{u},\tilde{v}))\, \underbrace{F(\tilde{u},\tilde{v})}_{\in S^2}$

$\Rightarrow \quad |f(F(u,v))\, \underbrace{F(u,v)}_{|F|=1}| = |f(F(\tilde{u},\tilde{v}))\, \underbrace{F(\tilde{u},\tilde{v})}_{|F|=1}|$

$\Rightarrow \quad f(F(u,v)) = f(F(\tilde{u},\tilde{v}))$

$\downarrow$

$F(u,v) = F(\tilde{u},\tilde{v})$

F is injective $\Rightarrow (u,v) = (\tilde{u},\tilde{v})$.

G is C^0 ✓

$G^{-1}(x,y,z) = ?$

let $\underbrace{G(u,v)}_{\text{want to find}} = (x,y,z) \in \Sigma \leftarrow$ given

$\Rightarrow \quad f(F(u,v))\, \underbrace{F(u,v)}_{\in S^2} = (x,y,z).$

$\overset{|\cdot|}{\Rightarrow} \quad f(F(u,v)) = \sqrt{x^2+y^2+z^2}.$

$$F(u,v) = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} \Rightarrow (u,v) = F^{-1}\left(\frac{x,y,z}{\sqrt{x^2+y^2+z^2}}\right).$$

$$G^{-1}(\underbrace{x,y,z}_{\in\Sigma}) = F^{-1}\left(\frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}}\right) \leftarrow C^0$$

because $(x,y,z)\in\Sigma$.

$\Rightarrow (x,y,z) \neq (0,0,0)$.

③ $G(u,v) = f(F(u,v))\,F(u,v)$

$$\frac{\partial G}{\partial u} = \underbrace{\frac{\partial}{\partial u}(f\circ F)}_{=:\frac{\partial f}{\partial u}} \cdot F(u,v) + f(F(u,v))\frac{\partial F}{\partial u}$$

$$\frac{\partial G}{\partial v} = \frac{\partial f}{\partial v} F(u,v) + f(F(u,v))\frac{\partial F}{\partial v}.$$

$$\frac{\partial G}{\partial u}\times\frac{\partial G}{\partial v} = f\frac{\partial f}{\partial u}\left(F\times\frac{\partial F}{\partial v}\right) + f\frac{\partial f}{\partial v}\left(\frac{\partial F}{\partial u}\times F\right) + f^2\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}.$$

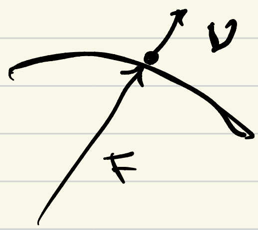
$$\left(\frac{\partial G}{\partial u}\times\frac{\partial G}{\partial v}\right)\cdot F = 0 + 0 + \underbrace{f^2}_{\neq 0}\underbrace{\left(\frac{\partial F}{\partial u}\times\frac{\partial F}{\partial v}\right)\cdot F}$$

$= \phi F$

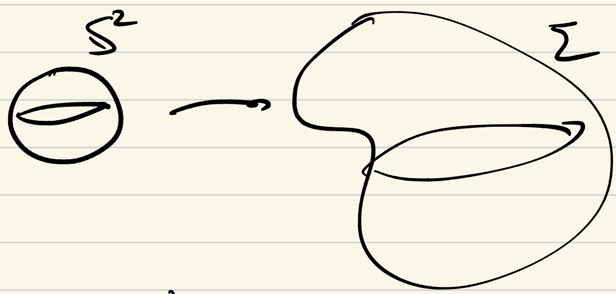
$\phi \neq 0$, F normal.

$$= \underbrace{f^2\phi}_{\neq 0}.$$

$$\Rightarrow \frac{\partial G}{\partial u}\times\frac{\partial G}{\partial v} \neq 0.$$



(b). $S^2 \xrightarrow{\Phi} \Sigma$ diffeomorphism $\overset{\text{def}}{\iff}$ Φ is bijective and Φ and Φ^{-1} are C^∞.



$x \in S^2 \mapsto f(\vec{x})\,\vec{x} \in \Sigma.$

Check:

- $G^{-1}\circ\Phi\circ F$ is C^∞

  G: para. of Σ; F: para. of S^2

- $F^{-1}\circ\Phi^{-1}\circ G$ is C^∞.

$$G^{-1}\circ\Phi\circ F(u,v) = G^{-1}\Big(f(F(u,v))\,F(u,v)\Big)$$

$$= F^{-1}\left(\frac{\cancel{f(F(u,v))}\,F(u,v)}{|f(F(u,v))\,F(u,v)|}\right)$$

$$= \cancel{f(F(u,v))}$$

$$= F^{-1}\big(F(u,v)\big)$$

$$= (u,v).$$

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(a) Suppose $F(u_1, u_2)$ is a C^∞ local parametrization of Σ.

i. Show that $\frac{\partial \nu}{\partial u_i}(p) \in T_p\Sigma$. ✓

$$\frac{\partial \nu}{\partial u_i} \cdot \nu = \frac{1}{2}\frac{\partial}{\partial u_i} \nu \cdot \nu = \frac{1}{2}\frac{\partial}{\partial u_i} 1 = 0.$$

ii. Consider the linear map $h : T_p\Sigma \to T_p\Sigma$ defined by:

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(a)(ii). $$h\left(\frac{\partial F}{\partial u_i}\right) = \frac{\partial \nu}{\partial u_i}$$

$$\left\langle h\left(\frac{\partial F}{\partial u_i}\right), \frac{\partial F}{\partial u_j}\right\rangle = \left\langle \frac{\partial \nu}{\partial u_i}, \frac{\partial F}{\partial u_j}\right\rangle$$

$$= \frac{\partial}{\partial u_i}\underbrace{\left\langle \overset{\text{normal}}{\nu}, \overset{\text{tangent}}{\frac{\partial F}{\partial u_j}}\right\rangle}_{\equiv 0} - \left\langle \nu, \frac{\partial^2 F}{\partial u_i \partial u_j}\right\rangle.$$

$$= -\left\langle \nu, \frac{\partial^2 F}{\partial u_i \partial u_j}\right\rangle$$

$$\left\langle \frac{\partial F}{\partial u_i}, h\left(\frac{\partial F}{\partial u_j}\right)\right\rangle = -\left\langle \nu, \frac{\partial^2 F}{\partial u_j \partial u_i}\right\rangle =$$

(b) $$\frac{\partial \hat{F}}{\partial u_1} = \frac{\partial F}{\partial u_1} + \frac{\partial f}{\partial u_1}\nu + f\, h\left(\frac{\partial F}{\partial u_1}\right) = (\text{id} + fh)\left(\frac{\partial F}{\partial u_1}\right) + \frac{\partial f}{\partial u_1}\nu$$

$$\frac{\partial \hat{F}}{\partial u_2} = \frac{\partial F}{\partial u_2} + \frac{\partial f}{\partial u_2}\nu + f\, h\left(\frac{\partial F}{\partial u_2}\right) = (\text{id} + fh)\left(\frac{\partial F}{\partial u_2}\right) + \frac{\partial f}{\partial u_2}\nu$$

$$\frac{\partial \hat{F}}{\partial u_1} \times \frac{\partial \hat{F}}{\partial u_2} = (\text{id} + fh)\left(\frac{\partial F}{\partial u_1}\right) \times (\text{id} + fh)\left(\frac{\partial F}{\partial u_2}\right)$$
$$+ (\text{id} + fh)\left(\frac{\partial F}{\partial u_1}\right) \times \frac{\partial f}{\partial u_2}\nu + \frac{\partial f}{\partial u_1}\nu \times (\text{id} + fh)\left(\frac{\partial F}{\partial u_2}\right)$$

$$\left(\frac{\partial \hat{F}}{\partial u_1} \times \frac{\partial \hat{F}}{\partial u_2}\right) \cdot \left((\text{id} + fh)\left(\frac{\partial F}{\partial u_1}\right) \times (\text{id} + fh)\left(\frac{\partial F}{\partial u_2}\right)\right)$$

linearly indep. ⇝ cross prod $\neq 0$.

$$= \left|(\text{id} + fh)\left(\frac{\partial F}{\partial u_1}\right) \times (\text{id} + fh)\left(\frac{\partial F}{\partial u_2}\right)\right|^2 + 0 + 0$$

invertible

linearly indep.