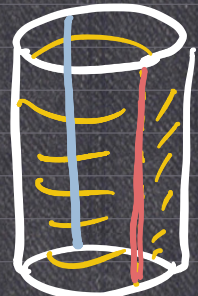
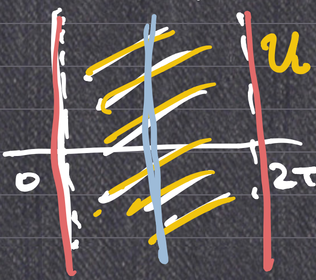
$x^2+y^2=1$

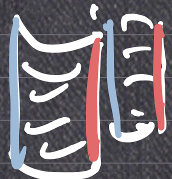


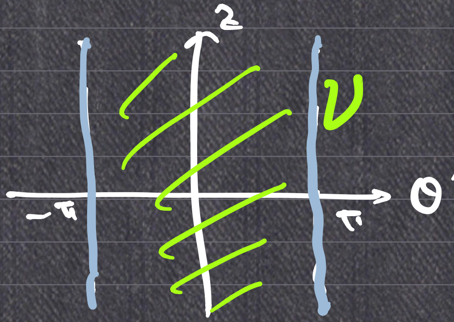
$F(\theta,z)$
$:= (\cos\theta, \sin\theta, z)$

$G(\theta', z')$
$= (\cos\theta', \sin\theta', z')$



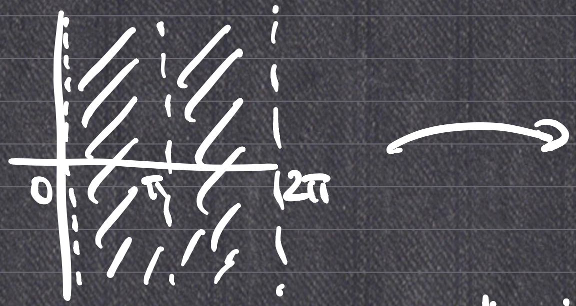
$F(U) \cap G(V)$
$=$





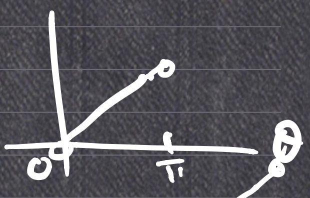
$\frac{\partial F}{\partial \theta} \times \frac{\partial F}{\partial z} \neq 0.$

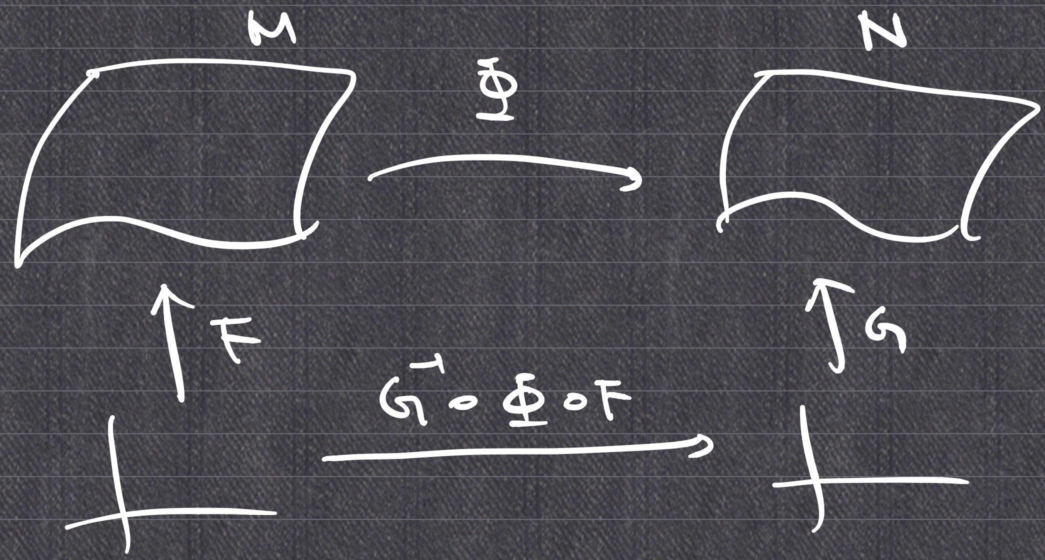
$G^{-1}\circ F :$



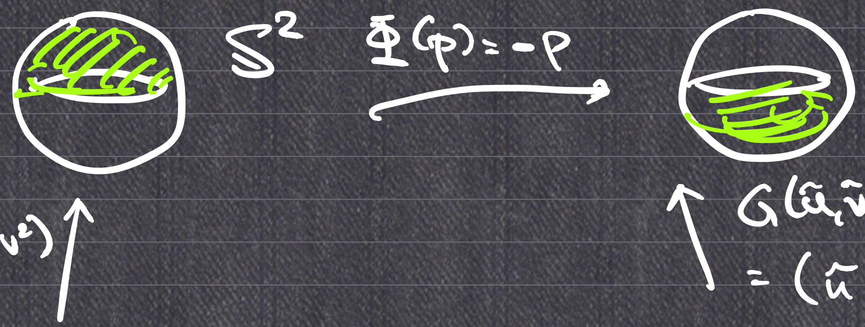
$G^{-1}\circ F(\theta, z) = \begin{cases} (\theta, z) & \text{if} \\ (\theta - 2\pi, z) & \text{if} \end{cases}$

$G^{-1}\circ F : ((0,\pi)\times\mathbb{R}) \sqcup ((\pi,2\pi)\times\mathbb{R}) \longrightarrow \ldots$





e.g.



$F(u,v) = (u, v, \sqrt{1-u^2-v^2})$

$G(\tilde{u},\tilde{v}) = (\tilde{u}, \tilde{v}, -\sqrt{1-\tilde{u}^2-\tilde{v}^2})$

$$G^{-1}\circ\Phi\circ F(u,v)$$

$$= G^{-1}\circ\Phi\left(u, v, \sqrt{1-u^2-v^2}\right)$$

$$= G^{-1}\left(\underbrace{-u, -v, -\sqrt{1-u^2-v^2}}\right) = (-u, -v).$$

$$G(\underset{-u,\ -v}{?}) = (-u, -v, -\sqrt{1-u^2-v^2})$$

$$\frac{\partial\Phi}{\partial u} := \frac{\partial}{\partial u}(\Phi\circ F)$$

$$= \frac{\partial}{\partial u}\left(-u, -v, -\sqrt{1-u^2-v^2}\right) = (-1, 0, * \ldots)$$

$F(u,v)$

$$\frac{\partial \Phi}{\partial \tilde{u}} = \frac{\partial}{\partial \tilde{u}}(\Phi \circ G)$$

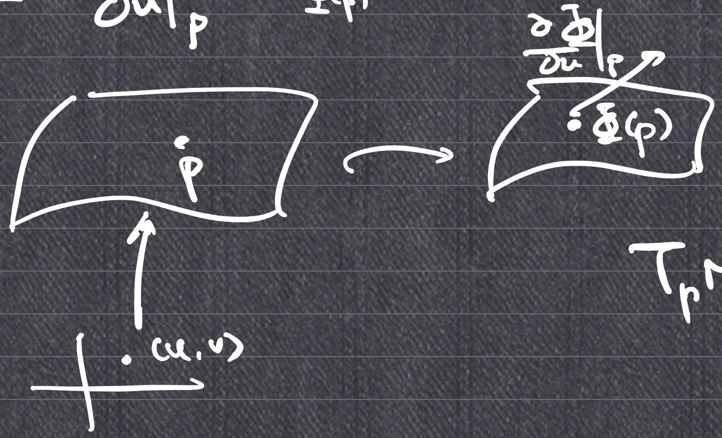
$$= \frac{\partial}{\partial \tilde{u}} \Phi\left(\tilde{u}, \tilde{v}, -\sqrt{1-\tilde{u}^2-\tilde{v}^2}\right)$$

$$= \frac{\partial}{\partial \tilde{u}}\left(-\tilde{u}, -\tilde{v}, \sqrt{1-\tilde{u}^2-\tilde{v}^2}\right)$$

$$= (-1, 0, -\circledast)$$

Claim. $\left.\frac{\partial \Phi}{\partial u}\right|_p \in T_{\Phi(p)} N$

$$\begin{array}{ccc} & & \Phi: M \to N \\ & F \nearrow & \uparrow G(\tilde{u}, \tilde{v}) \end{array}$$



$$T_p M = \operatorname{span}\left\{\frac{\partial F}{\partial u}(p), \frac{\partial F}{\partial v}(p)\right\}.$$

Proof: $\frac{\partial \Phi}{\partial u} = \frac{\partial}{\partial u}(\Phi \circ F) = \frac{\partial}{\partial u}\Big(\underbrace{G}_{\mathbb{R}^2 \to \mathbb{R}^3} \circ \underbrace{(G^{-1} \circ \Phi \circ F)}_{\mathbb{R}^2 \to \mathbb{R}^2}\Big)$

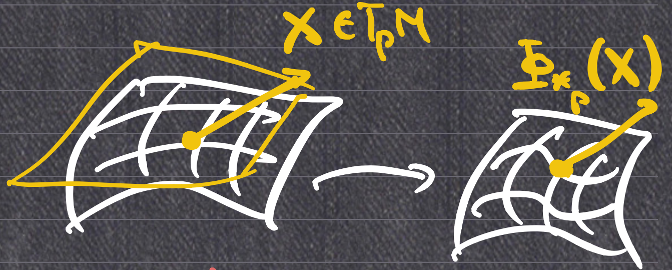
$$= \frac{\partial \tilde{u}}{\partial u}\frac{\partial G}{\partial \tilde{u}} + \frac{\partial \tilde{v}}{\partial u}\frac{\partial G}{\partial \tilde{v}} \in T_{\Phi(p)} N.$$

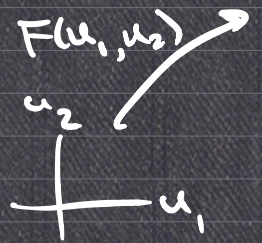
∎

$\Phi: M \to N$, $p \in M$

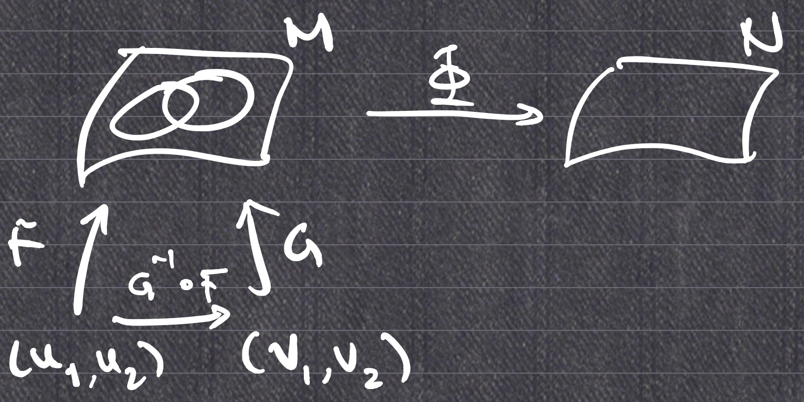
$\Phi_{*p}: T_pM \to T_{\Phi(p)}N$

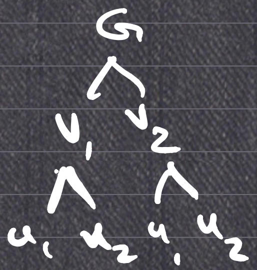
linear map.





$$\Phi_{*p}\left(\frac{\partial F}{\partial u_i}(p)\right) := \frac{\partial \Phi}{\partial u_i}.$$





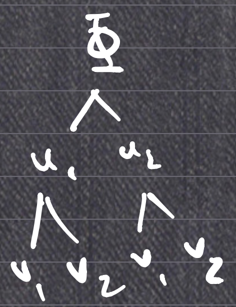
WANT: $\Phi_{*p}\left(\frac{\partial G}{\partial v_j}\right) = \frac{\partial \Phi}{\partial v_j}$

Proof:

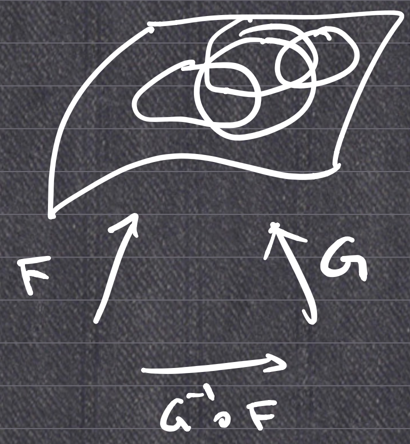
$$\Phi_*\left(\frac{\partial G}{\partial v_j}\right) = \Phi_*\left(\frac{\partial u_1}{\partial v_j}\overset{\frac{\partial F}{\partial u_1}\,=}{\frac{\partial G}{\partial u_1}} + \frac{\partial u_2}{\partial v_j}\overset{\frac{\partial F}{\partial u_2}\,\parallel}{\frac{\partial G}{\partial u_2}}\right)$$

$$= \frac{\partial u_1}{\partial v_j}\frac{\partial \Phi}{\partial u_1} + \frac{\partial u_2}{\partial v_j}\frac{\partial \Phi}{\partial u_2}$$

$$= \frac{\partial \Phi}{\partial v_j}$$



Summary:



- $G^{-1} \circ F$ smooth
- tangent plane

  $T_pM := \text{span}\left\{\frac{\partial F}{\partial u}(p), \frac{\partial F}{\partial v}(p)\right\}$
- $\frac{\partial \Phi}{\partial u} :=$ ✓
- $\Phi_*\left(\frac{\partial F}{\partial u}\right) :=$ / .

$\underbrace{\text{Abstract}}$ manifold

do not assume $M \subset \underbrace{\mathbb{R}^N}_{\text{ambient}}$.

M topological space

layman: can make sense of open sets.

Hausdorff

second countable.

admits a countable basis for the open sets.

Def (Topological manifold)

M is a n-dim topological manifold

$\overset{\text{def}}{\iff}$ M is a topological space, Hausdorff, 2nd countable, and

$\forall p \in M$, $\exists$ homeomorphism $F: U \xrightarrow{\subset \mathbb{R}^n} O \subset M$. $\quad$ ($p \in O$)

Def (smooth manifold).

M is an n-dim smooth manifold

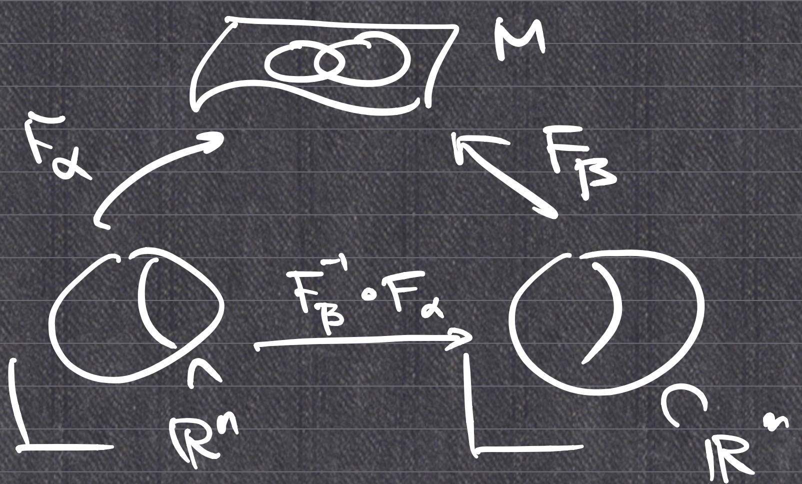
$\overset{\text{def}}{\iff}$ M is an n-dim'l topologial manifold

with $\{F_\alpha : U_\alpha \to O_\alpha\}$ homeomorphisms s.t. ($O_\alpha \subset M$)

① $\bigcup_\alpha F_\alpha(U_\alpha) = M$

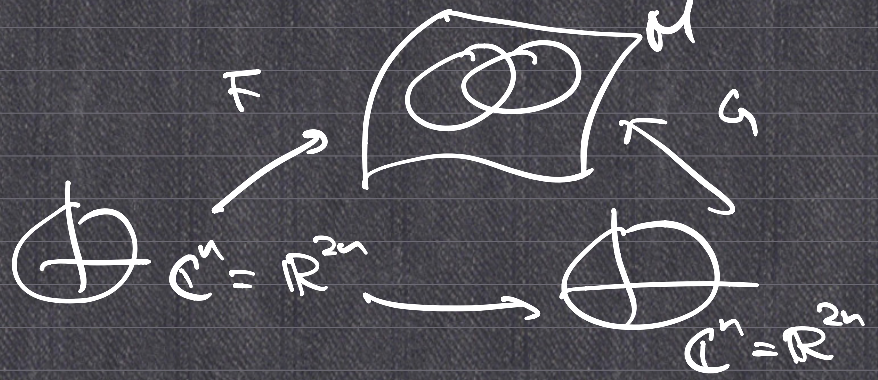
② $F_\beta^{-1} \circ F_\alpha$ is smooth (C^∞)

$\forall \alpha, \beta$ s.t. $F_\alpha(U_\alpha) \cap F_\beta(U_\beta) \neq \emptyset$.



FYI: $C^\infty \to C^k$: then call C^k-manifold.

FYI:



$G^{-1} \circ F$ is holomorphic $\forall F, G$.

$\rightarrow$ say M is a complex manifold.

---

examples

① $\mathbb{R}^n$ $\qquad F : \mathbb{R}^n \to \mathbb{R}^n$

$F = id$.

② All regular surfaces in $\mathbb{R}^3$.

③ $M = \mathbb{C} \cup \{\infty\} = \{x+yi : x, y \in \mathbb{R}\} \cup \{\infty\}$.

$$\mathbb{R}^2 \xrightarrow{F} M \xleftarrow{G} \mathbb{R}^2$$

$$G(x,y) = \begin{cases} \dfrac{1}{x+yi} & \text{if } (x,y) \neq (0,0) \\ \infty & \text{if } (x,y) = (0,0) \end{cases}$$

$F(x,y) = x+yi$

$$G^{-1} \circ F(x,y) = G^{-1}(x+yi)$$

$$= \frac{1}{x+yi} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$$

$G(\ ?\ ) = x+yi$ $\qquad = \left(\underbrace{\frac{x}{x^2+y^2}}, \ \underbrace{-\frac{y}{x^2+y^2}}\right)$

$G\left(\frac{1}{x+yi}\right) = x+yi$ $\qquad$ is C^∞ on $\boxed{\mathbb{C}\setminus\{0\}}$.

$G^{-1}\circ F : \mathbb{C}\setminus\{0\} \longrightarrow \mathbb{C}\setminus\{0\}$.

$F^{-1}\circ G$ ✓