**Prop:** Given $g(x,y,z): \mathbb{R}^3 \to \mathbb{R}$, $C^\infty$

and consider $\Sigma := g^{-1}(c)$

$$= \underbrace{\{(x,y,z) : g(x,y,z) = c\}}_{\text{Assume } \neq \emptyset}.$$

If $\nabla g(x,y,z) \neq \vec{0}$

$\forall (x,y,z) \in \Sigma = g^{-1}(c)$.

then $\Sigma$ is a regular surface.

e.g. $\{\underbrace{x^2+y^2}_{g}+z^2 = 1\} = S^2$

$\nabla g = (2x, 2y, 2z) = \vec{0}$

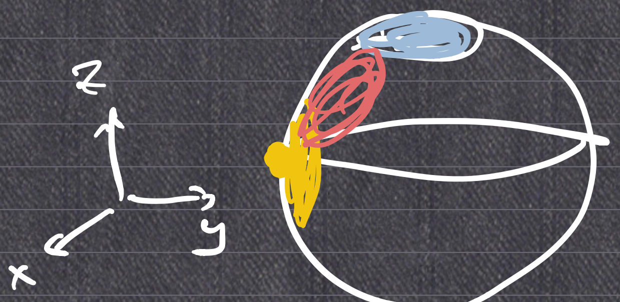
$\Leftrightarrow (x,y,z) = (0,0,0) \notin S^2$.

$\nabla g(P) \neq \vec{0} \quad \forall P \in S^2$

$\Rightarrow$ by Prop., $S^2$ is a regular surface.

**HW:**

$\boxed{\{\vec{x} : \vec{x}^T A \vec{x} = 1\}}$




**Proof:**

Implicit Function Theorem

$\nabla g(P) \neq \vec{0} \Rightarrow \frac{\partial g}{\partial z}(P) \neq 0 \Rightarrow z = f_1(x,y)$

or $\frac{\partial g}{\partial y}(P) \neq 0 \Rightarrow y = f_2(x,z)$

or $\frac{\partial g}{\partial x}(P) \neq 0 \Rightarrow x = f_3(y,z)$.

$\{g(x,y,z)=c\}$

tangent plane at $\underbrace{(x_0,y_0,z_0)}_{P_0} \in g^{-1}(c)$ :

$$\left.\frac{\partial g}{\partial x}\right|_{P_0} \cdot (x-x_0) + \left.\frac{\partial g}{\partial y}\right|_{P_0} (y-y_0) + \left.\frac{\partial g}{\partial z}\right|_{P_0} (z-z_0) = 0.$$

If $\neq 0$.

$\Rightarrow$ $z = \text{- - - - -}$ .

$(x,y)$ only.


$z = $ fct of $(x,y)$

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**Prop:** $M = g^{-1}(c)$ where

$\nabla g(P) \neq 0 \quad \forall P \in M$.

let

$F(u,v) : \mathcal{U} \to \mathcal{O} \subset M$ satisfy : ($\mathcal{U} \subset \mathbb{R}^2$)

① $F$ is $C^\infty$ on $\mathcal{U}$

② $F$ is bijective (and $F$ is continuous)

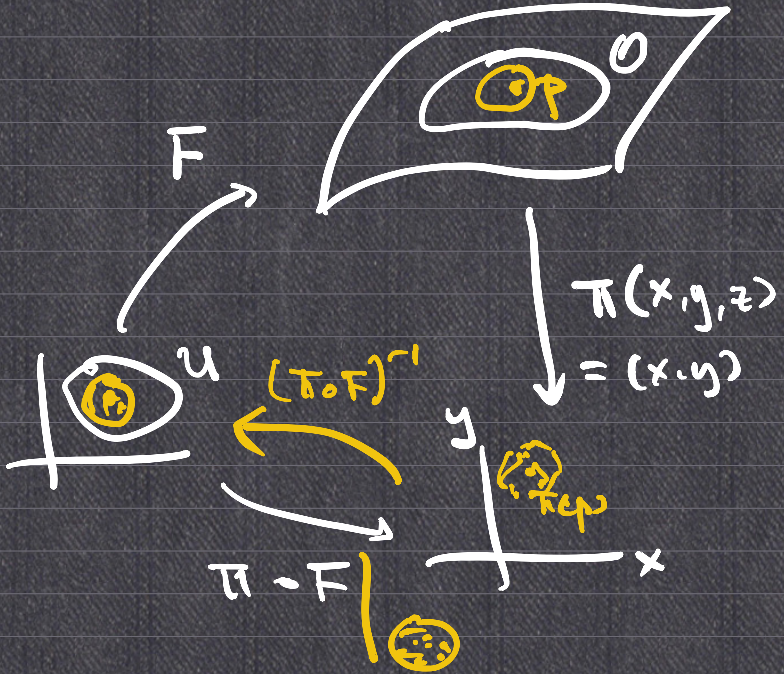
③ $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq \vec{0}$ on $\mathcal{U}$.

Then $F^{-1} : \mathcal{O} \to \mathcal{U}$ is continuous.

$F(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$

$r = \sqrt{x^2+y^2}$

| | |
|---|---|
| $\tan^{-1}\frac{y}{x} + \frac{\pi}{2}$ | $\tan^{-1}\frac{y}{x}$ |
| $\tan^{-1}\frac{y}{x} - \frac{\pi}{2}$ | $\tan^{-1}\frac{y}{x}$ |




$F(u,v)$

$= (x(u,v), y(u,v), z(u,v))$

$$0 \neq \frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$

Assume WLOG

$$\underbrace{\left(\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v}\right) \cdot \hat{k} \neq 0.}$$

$$\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} \neq 0$$

$$\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} \neq 0.$$

$\pi \circ F(u,v) = (x(u,v), y(u,v))$

IFT $\Rightarrow$ $\boxed{\text{locally}}$ $(\pi\circ F)^{-1}$ exists.

and $(\pi\circ F)^{-1}$ is $C^\infty$.

$$F^{-1} = \underbrace{(\pi\circ F)^{-1}}_{\text{cts}} \circ \underbrace{\pi}_{\text{cts}} \quad \Uparrow \text{ cts}$$

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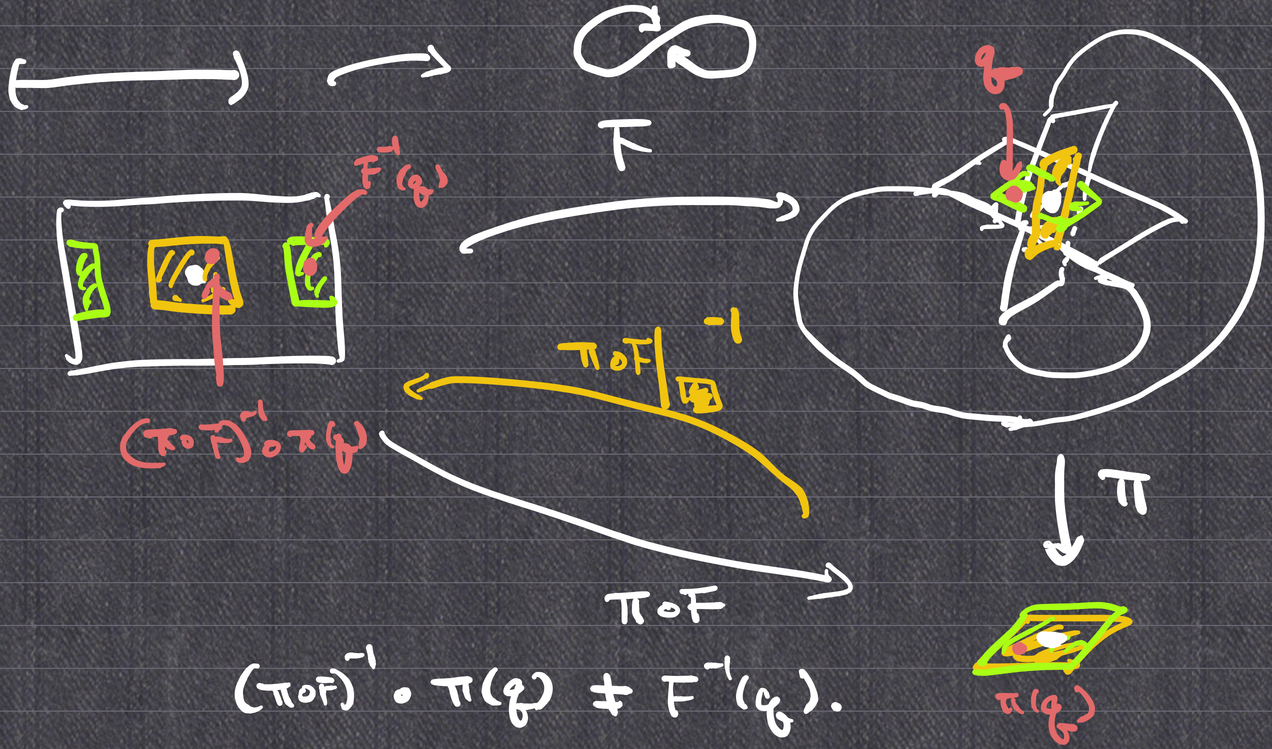
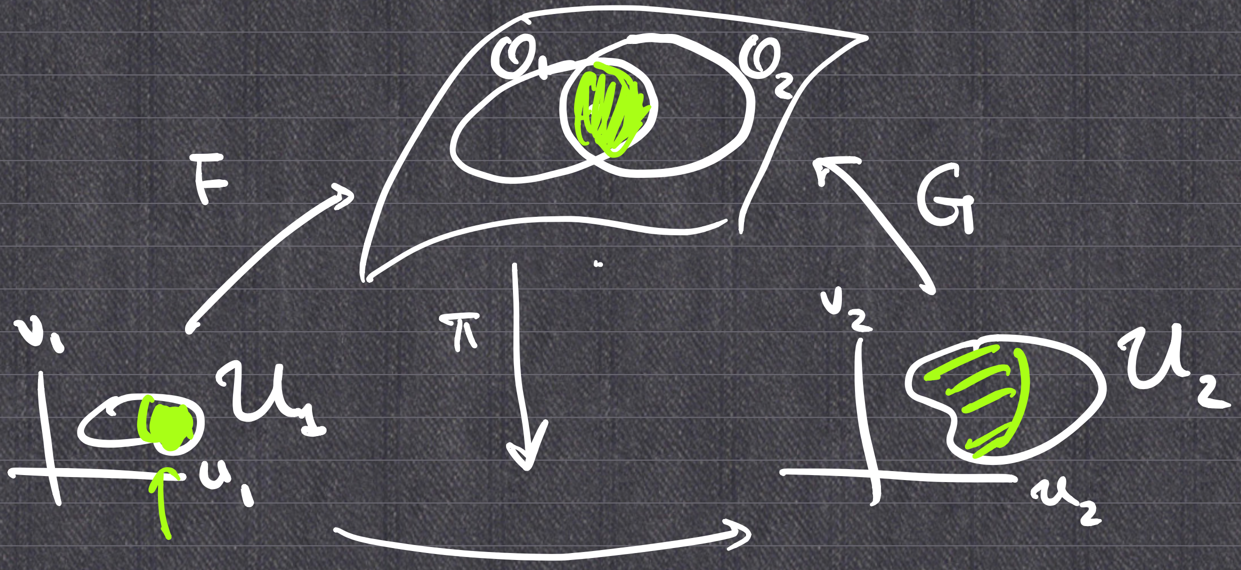



$(\pi\circ F)^{-1} \circ \pi(q) \neq F^{-1}(q).$




$$\underbrace{G^{-1}\circ F\big|_{F^{-1}(\mathcal{O}_1\cap\mathcal{O}_2)}}_{\text{transition map between } G \text{ and } F.}$$

$$\frac{\partial G}{\partial u_2}\times\frac{\partial G}{\partial v_2}\neq 0 \text{ at } P \quad \overset{\text{WLOG}}{\rightsquigarrow} \quad \text{Assume } \left(\frac{\partial G}{\partial u_2}\times\frac{\partial G}{\partial v_2}\right)\cdot\hat{k}\neq 0 \text{ at } P.$$

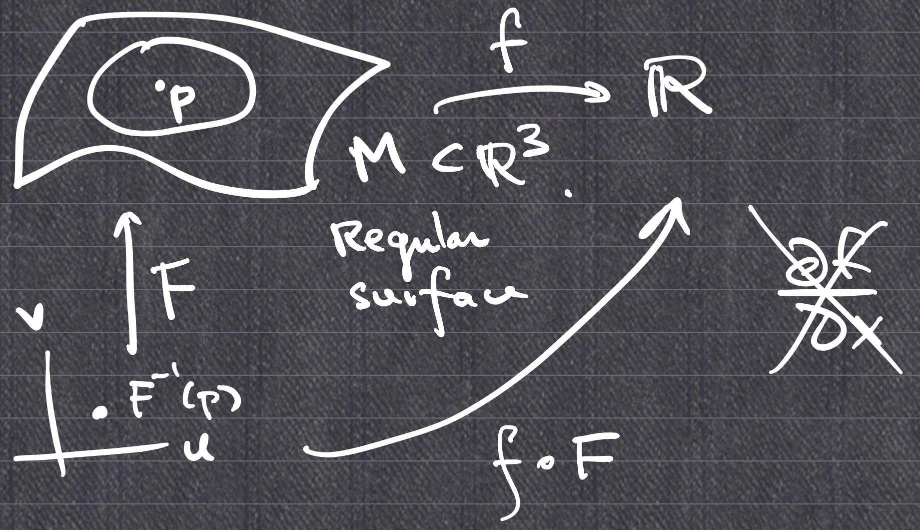
$\pi(x,y,z)=(x,y)$

$$\Rightarrow \det D(\pi\circ G)\neq 0 \text{ at } P$$

$\pi\circ G$ is locally invertiable near $P$.

and $(\pi\circ G)^{-1}$ is $C^\infty$.

$$G^{-1}\circ F = \underbrace{(\pi\circ G)^{-1}}_{C^\infty}\circ\underbrace{(\pi\circ F)}_{C^\infty} \text{ is } C^\infty.$$

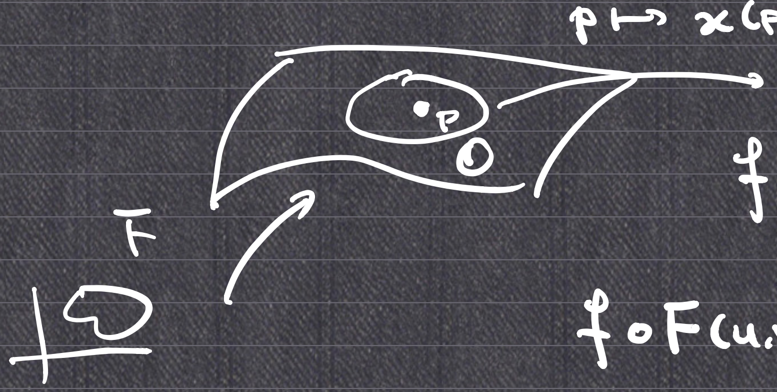



$$\frac{\partial f}{\partial u}(p) := \frac{\partial}{\partial u}(f\circ F)\Big|_{F^{-1}(p)}.$$

$f$ is $C^k$ at $p\in M \overset{\text{def}}{\iff} f\circ F$ is $C^k$ at $F^{-1}(p)\in\mathbb{R}^2$.

$$f\circ G = \underbrace{(f\circ F)}_{\text{if } C^\infty}\circ\underbrace{(F^{-1}\circ G)}_{C^\infty}. \text{ is } C^\infty.$$

Given $F(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$ is $C^\infty$.




$f(p) := x(p)$

$$f\circ F(u,v) = f\begin{pmatrix} x(u,v),\ y(u,v) \\ z(u,v)\end{pmatrix}$$

$$= \underbrace{x(u,v)}_{C^\infty}.$$

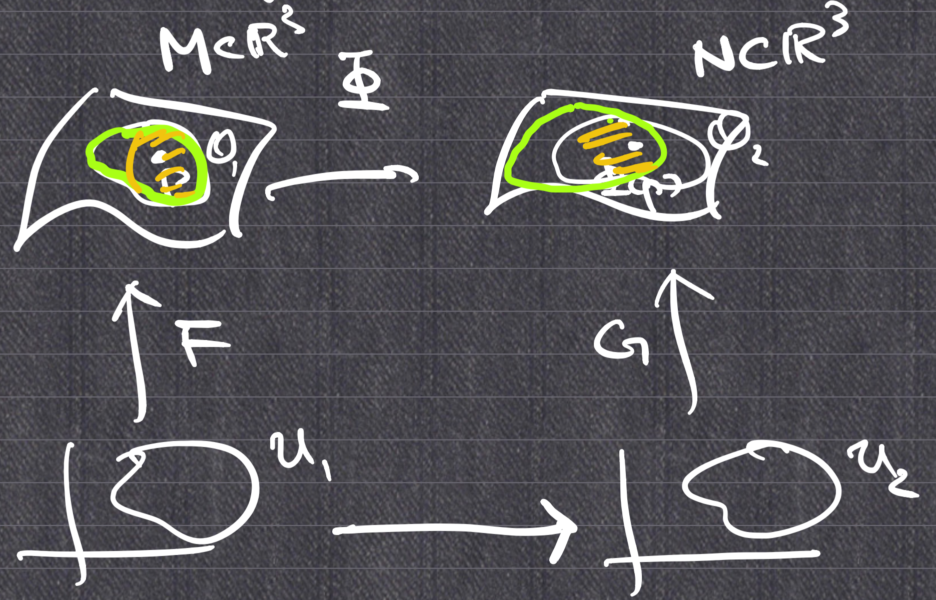
$\Rightarrow f$ is $C^\infty$ on $\mathcal{O}$.

e.g. $f(p) = x(p)^2 + y(p)^2 + z(p)^2$ is $C^\infty$.

$p \in M$

$$f \circ F(u,v) = x(u,v)^2 + y(u,v)^2 + z(u,v)^2.$$

M, N regular surfaces:




$\Phi$ is $C^k$ at $p \in M$

$\overset{\text{def}}{\iff}$ $G^{-1} \circ \Phi \circ F \big|_{F^{-1}(F(\mathcal{U}_1) \cap \Phi^{-1}(G(\mathcal{U}_2)))}$ is $C^k$ at $F^{-1}(p)$.