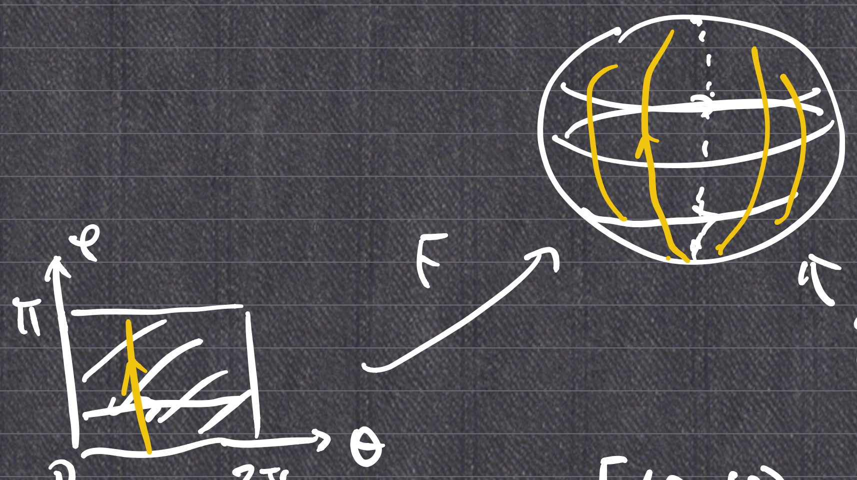
§1. Regular surfaces



$$\underbrace{x^2+y^2+z^2}_{g(x,y,z)}=1$$

$$g(x,y,z) = x^2+y^2+z^2.$$

$g^{-1}(1)$.

$F(\theta,\varphi) : (0,2\pi)\times(0,\pi) \to$ sphere.

$$= (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)$$

__Def:__ (C^k- local parametrization)

Given $M \subset \mathbb{R}^3$, $\quad F : \underset{\text{open}}{U \subset \mathbb{R}^2} \to \underset{\text{open}}{O \subset M}$. is called

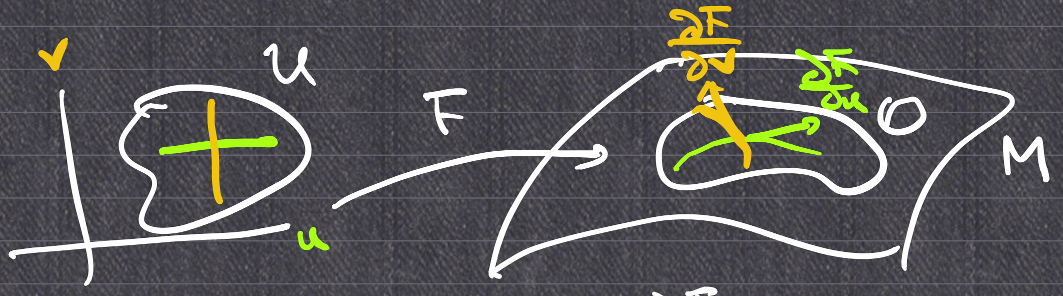
a C^k local parametrisation $\overset{\text{def}}{\Longleftrightarrow}$

① $F : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is C^k on U $\qquad F(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$

② $F : U \to O$ is homeomorphism.
(F is bijective, F is continuous and $F^{-1} : O \to U$ is continuous)

③ $\dfrac{\partial F}{\partial u} \times \dfrac{\partial F}{\partial v} \neq 0 \quad \forall (u,v) \in U.$ $\quad \left(\Leftrightarrow \left\{ \dfrac{\partial F}{\partial u}, \dfrac{\partial F}{\partial v} \right\} \text{ linearly indep.} \right)$

tangent vectors



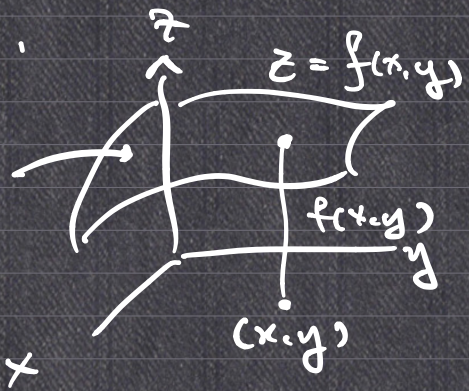
$$\operatorname{span}\left\{\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v}\right\}.$$

e.g. $f(x,y): \mathbb{R}^2 \to \mathbb{R}$, C^k on $\mathbb{R}^2$.

$$\Gamma := \{(x,y,f(x,y)) : (x,y) \in \mathbb{R}^2\}.$$

$$F(u,v) = (u,v,f(u,v)) : \underbrace{\mathbb{R}^2}_{U} \to \underbrace{\Gamma}_{O}$$

Γ



① ✓

② $F^{-1}(x,y,z) = (x,y)$.

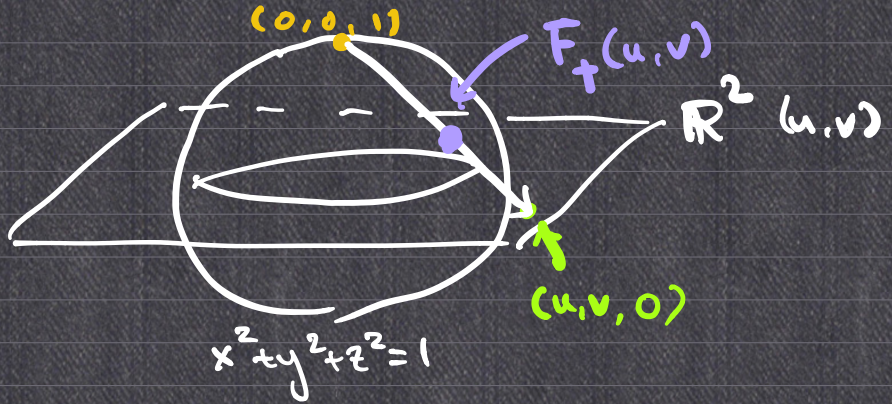
continuous ✓

③ $$\frac{\partial F}{\partial u} = \left(1, 0, \frac{\partial f}{\partial u}\right)$$

$$\frac{\partial F}{\partial v} = \left(0, 1, \frac{\partial f}{\partial v}\right)$$

$$\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} = (*, *, 1) \neq 0$$ ✓

$\therefore$ F is a C^k local parametrization of Γ.



$$\vec{r}(t) = (0,0,1) + t\big((u,v,0)-(0,0,1)\big)$$

Find t s.t. $|\vec{r}(t)| = 1$. $\longrightarrow$ plug into $\vec{r}(t)$.

$$F_+(u,v) = \left(\frac{2u}{u^2+v^2+1},\ \frac{2v}{u^2+v^2+1},\ \frac{u^2+v^2-1}{u^2+v^2+1}\right).$$

$$F_+(u,v) : \underbrace{\mathbb{R}^2}_{\mathcal{U}} \longrightarrow \underbrace{S^2 \setminus \{(0,0,1)\}}_{O} \subset S^2.$$

① F_+ is C^∞ on $\mathbb{R}^2$.

② F_+ injective (Exercise)

$(x,y,z) \in S^2 \setminus \{(0,0,1)\}$

find $F_+(\ ?\ ) = (x,y,z)$.

$F_+\left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ // (surjective)

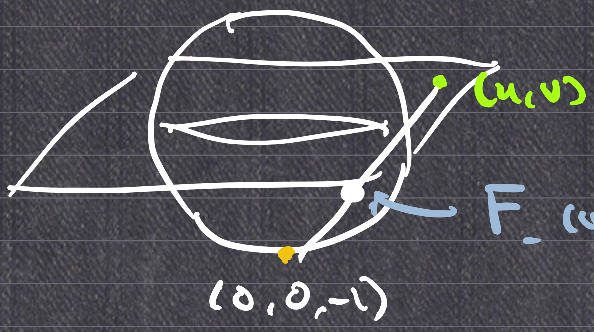
$$F_+^{-1}(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \quad \text{continuous}$$

on $O \longrightarrow \mathbb{R}^2$.

$z \neq 1$.

③ $\frac{\partial F}{\partial u} \times \frac{\partial F}{\partial v} \neq 0$ (Exercise).

F_+ is C^∞ local parametrization of S^2.

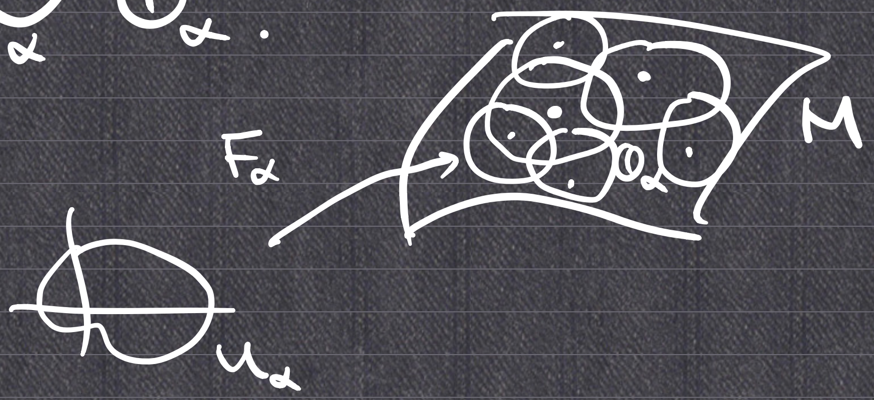


$F_-(u,v) = \underbrace{(\,?\,,\,?\,,\,?\,)}_{\text{Exercise.}}$

Definition: (C^k-surface)

$M \subset \mathbb{R}^3$ is called a C^k-surface if $\exists \{F_\alpha : \mathcal{U}_\alpha \to \mathcal{O}_\alpha\}$ a family of C^k-local parametrizations such that

$$M = \bigcup_\alpha \mathcal{O}_\alpha.$$



In particular,

C^∞-surface

=: regular surface.