MATH 4033 • Spring 2021 • Calculus on Manifolds
Problem Set #1 • Regular Surfaces • Due Date: 28/02/2019, 11:59PM

Instructions: *You can form a group of 1 to 2 students to work on the homework, and submit one copy of your solution to Canvas. All students in the same group will receive the same score. Grouping can be different in different homework. The problem sets are typically more challenging than many exercises in the lecture note. If you get stuck, it is best to first complete some related exercises in the lecture notes before trying the problem sets (as a corollary, start* ***early*** *working on the problem sets).*

1. (10 points) Recall two facts in MATH 2131:
   - An $n \times n$ symmetric matrix A is said to be **positive semi-definite** if $\sum_{i,j=1}^{n} A_{ij}x_i x_j \geq 0$ for any $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$.
   - (Spectrum Theorem) Any symmetric matrix A is orthogonally diagonalizable, i.e. there exist $P \in O(n)$ and a diagonal matrix D such that $A = PDP^T$.

   (a) Show that A is positive semi-definite if and only if all eigenvalues of A are non-negative.

   (b) Show that for a positive semi-definite matrix A, we have $\sum_{i,j=1}^{n} A_{ij}x_i x_j = 0$ if and only if $x \in \ker(A)$.

2. (25 points) Let A be a 3×3 symmetric real matrix. Consider the set
$$\Sigma := \{x \in \mathbb{R}^3 : x^T A x = 1\}.$$
   Here $x \in \mathbb{R}^3$ is regarded as a column vector.

   (a) Show that Σ is a regular surface whenever $\Sigma \neq \emptyset$.

   (b) Suppose A is positive semi-definite (i.e. eigenvalues are non-negative). Show that Σ (whenever non-empty) is diffeomorphic to either a sphere, a cylinder, or a pair of planes.

3. (30 points) Fix $\varphi \in (0, \pi/2)$, and consider the circles:
$$\begin{aligned}\gamma_+ &= (\cos(u+\varphi), \sin(u+\varphi), 1)\\ \gamma_- &= (\cos(u-\varphi), \sin(u-\varphi), -1)\end{aligned}$$
   Define a parametrization $F(u,v) : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ by
$$F(u,v) := (1-v)\gamma_-(u) + v\gamma_+(u).$$

   (a) Show that the image of F is part of a hyperboloid $\Sigma(\varphi)$:
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
   for some $a, b, c > 0$ depending only φ.

   (b) Show that $F(u,v)$ is a C^∞ local parametrization of $\Sigma(\varphi)$. Verify all conditions (including F being a homeomorphism).

   (c) Construct another similar C^∞ local parametrization $G(u,v)$ of $\Sigma(\varphi)$ such that the images of F and G cover the whole $\Sigma(\varphi)$. Compute the transition map $G^{-1} \circ F$ explicitly – clearly state its domain.

4. (35 points) Let F_+ and F_- be the stereographic parametrizations of the unit sphere $\mathbb{S}^2$ as discussed in Example 1.5 of the lecture notes. Here we regard $\mathbb{C}$ as $\mathbb{R}^2$ by identifying $z = u + iv \in \mathbb{C}$ with $(u, v) \in \mathbb{R}^2$. Then, $F_+ : \mathbb{C} \to \mathbb{S}^2\backslash\{(0,0,1)\}$ and its inverse can be expressed as:

$$F_+(z) = \left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{2\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \qquad F_+^{-1}(x_1, x_2, x_3) = \frac{x_1 + x_2 i}{1 - x_3}$$

Here we use (x_1, x_2, x_3) for coordinates of $\mathbb{R}^3$ instead of (x, y, z) to avoid notation conflicts.

(a) Consider the south-pole stereographic parametrization $F_- : \mathbb{C} \to \mathbb{S}^2\backslash\{(0,0,-1)\}$. Find the explicit expressions of $F_-(z)$, where $z \in \mathbb{C}$, and $F_-^{-1}(x_1, x_2, x_3)$, where $(x_1, x_2, x_3) \in \mathbb{S}^2\backslash\{(0,0,-1)\}$.

(b) Verify that $F_-^{-1} \circ F_+(z) = \frac{1}{\bar{z}}$ and $F_+^{-1} \circ F_-(z) = \frac{1}{\bar{z}}$. Hence, by modifying one of F_+ and F_-, find a pair of C^∞ local parametrizations $\{G_+, G_-\}$ of $\mathbb{S}^2$ such that $G_-^{-1} \circ G_+(z) = \frac{1}{z}$ and $G_+^{-1} \circ G_-(z) = \frac{1}{z}$

(c) Consider the complex-valued function $f(z) = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}\backslash\{0\}$ such that $\alpha\delta \neq \beta\gamma$. Define a map $\Phi : \mathbb{S}^2 \to \mathbb{S}^2$ by:

$$\Phi(p) := \begin{cases} F_+(\alpha/\gamma) & \text{if } p = (0,0,1) \\ (0,0,1) & \text{if } p = F_+(-\delta/\gamma) \\ F_+ \circ f \circ F_+^{-1}(p) & \text{otherwise} \end{cases}$$

It can be checked that Φ is bijective (no need to show the detail).

i. Find an explicit expression of each of the following:

$$G_+^{-1} \circ \Phi \circ G_+(z) \quad G_-^{-1} \circ \Phi \circ G_+(z) \quad G_+^{-1} \circ \Phi \circ G_-(z) \quad G_-^{-1} \circ \Phi \circ G_-(z)$$

State the domain of each of them.

ii. Show that Φ is smooth at the point $(0,0,1)$.

iii. Express the tangent map Φ_* at $(0,0,1)$ in matrix representation and show that it is invertible.