香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

FINAL EXAMINATION

Course Code:	MATH 4033
Course Title:	Calculus on Manifolds
Semester:	Spring 2018-19
Date and Time:	12:30PM-3:30PM, 25 May 2019

Instructions

- Do **NOT** open the exam until instructed to do so.
- All mobile phones and communication devices should be switched **OFF**.
- It is an **OPEN-NOTES** exam. Authorized reference materials are the instructor's lecture notes, homework solutions, your own notebooks. No other reference materials (such as books) are allowed.
- Answer **ALL** problems. In **Part A**, write your answers **in the spaces provided**; whereas in **Part B**, write your solutions in the **yellow book**.
- You must **SHOW YOUR WORK** and **JUSTIFY YOUR STEPS** to receive credits in every problem in Part B.
- Some problems in Part B are structured into several parts. You can quote the results stated in the preceding parts to do the next part.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

"I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

Student's Signature: ______________________________

Student's Name: ______________________________
FAMILY NAME, First Name

HKUST ID: ____________________ **Seat Number:** ______________

Part A - Short Questions (25 points)

[Recommended time: $<$ 30 min.]
Instruction: Write your answers in **this question paper** in Part A.

1. Among the mathematicians listed below, who lived the longest? Put ✓ in the correct answer: **[2]**

   - ◯ Shiing-Shen Chern
   - ◯ Michael Atiyah
   - ✓ Leopold Vietoris (lived as long as his sequence)
   - ◯ Évariste Galois
   - ◯ Bernhard Riemann

2. Is it always true that $\omega \wedge \omega = 0$ for any differential form ω in $\mathbb{R}^2$? If true, explain briefly why. If false, give a simple counter-example. **[2]**

   Always true. If ω is a 1-form, then
   $\omega \wedge \omega = (-1)^{1 \times 1} \omega \wedge \omega = -\omega \wedge \omega$
   $\Rightarrow \omega \wedge \omega = 0$.
   If ω is a k-form with $k \geq 2$, then $\omega \wedge \omega$ is a $2k$-form, but $2k \geq 4 > 2 = \dim \mathbb{R}^2$.
   $\omega \wedge \omega = 0$

3. Write down a smooth 1-form on $\mathbb{R}^2 \backslash \{(1, 2)\}$ which is closed but not exact, or prove that such an example does not exist. **[2]**

   $$\frac{-(y-2)\,dx + (x-1)\,dy}{(x-1)^2 + (y-2)^2}$$

4. Let V be a vector space, and W be a subspace of V. Define EACH of the linear maps below so that the sequence becomes exact. **[3]**

   $$0 \to W \to V \to V/W \to 0$$

   $0 \to W$: $0 \mapsto 0$
   $W \to V$: $w \mapsto w$
   $V \to V/W$: $v \mapsto [v]$
   $V/W \to 0$: $[v] \mapsto 0$

5. Let $f : \mathbb{R}^{2019} \to \mathbb{R}$ be a smooth function. Suppose $\Sigma := f^{-1}(0)$ is non-empty and f is a submersion at every $p \in \Sigma$. Which of the following must be true? Put ✓ in ALL correct answer(s): **[4]**

   - ◯ Σ is a 2019-dimensional smooth manifold.
   - ✓ Σ is a smooth submanifold of $\mathbb{R}^{2019}$.
   - ◯ Σ is compact.
   - ✓ For any $p \in \Sigma$, there exist local coordinates $(u_1, \cdots, u_n)$ such that under the coordinates we have

$$[f_{*p}] = [1\ 0\ 0\ \cdots\ 0].$$

(Remark: consider $\frac{\nabla f}{|\nabla f|}$ in $\mathbb{R}^{2019}$, pick a basis of $\mathbb{R}^{2019}$ $\{e_1, \ldots, e_{2019}\}$ with $e_1 = \nabla f / |\nabla f|$.)

6. Which of the following statement(s) is/are always true? Put ✓ in ALL correct answer(s): **[6]**

   - ✓ The cotangent bundle of the tangent bundle of the Klein bottle is orientable.
   - ✓ If M and N are orientable smooth manifolds, then so is $M \times N$.
   - ◯ Suppose M is an orientable smooth manifold, and $N := M/\sim$ is a quotient set of M so that it is also a smooth manifold. Then N is also orientable.

     $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$
   - ◯ Let M be a smooth n-dimensional manifold (where $n \geq 2$), and let $p \in M$. Then as a vector space the dimension of $\wedge^2 T_p^* M$ is n^2.
   - ✓ Let $f : M \to \mathbb{R}$ be a smooth function from a smooth manifold M. Suppose $c \in \mathbb{R}$ such that $\Sigma := f^{-1}((-\infty, c])$ is non-empty, and suppose f is a submersion at any $p \in f^{-1}(c)$. Then Σ is a manifold with boundary.
   - ✓ Let I_i, $i = 1, \ldots, n$, be non-empty open intervals of $\mathbb{R}$ (possibly with different length). Then the 1st Betti number of the set

$$U := \underbrace{I_1 \times \cdots \times I_n}_{\text{star-shaped.}} \subset \mathbb{R}^n$$

     is equal to 0.

7. Based on the proof discussed in the lecture note or in class, which of the following is/are consequence(s) of the Inverse/Implicit Function Theorem for Euclidean spaces? **[6]**

   [Remark: If (A) is used to prove (B), and (B) is used to prove (C), then (C) is also regarded as a consequence of (A).]

   - ✓ Submersion Theorem
   - ◯ Cartan's Magic Formula
   - ✓ Regular surfaces in $\mathbb{R}^3$ are smooth manifolds.
   - ✓ Inverse Function Theorem for manifolds
   - ◯ $d^2 = 0$
   - ◯ Zigzag's Lemma

Part B - Long Questions (75 points): Answer ALL THREE problems

[Recommended time: Q1 $<$ 30 min; Q2 $<$ 1hr; Q3 $<$ 1hr]
Instruction: Write your solutions in the YELLOW answer book.

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1. Consider two C^∞ scalar functions $f, g : \mathbb{R}^{n\geq 3} \to \mathbb{R}$, and their non-empty level sets $\Sigma_f := f^{-1}(0)$ and $\Sigma_g := g^{-1}(0)$. Suppose $p \in \Sigma_f \cap \Sigma_g$ is a point such that $\{\nabla f(p), \nabla g(p)\}$ are linear independent vectors in $\mathbb{R}^n$.
   (a) Show that $\Sigma_f \cap \Sigma_g$ is locally a C^∞ manifold near p, i.e. there exists an open set U in $\mathbb{R}^n$ containing p such that $\Sigma_f \cap \Sigma g \cap U$ is a C^∞ manifold. What is its dimension? **[10]**
   (b) Show also that the set $\Sigma_f \cap \Sigma_g \cap U$ in (a) is a submanifold of Σ_f. **[5]**

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2. Consider a C^∞-manifold M^n which is compact, connected, orientable, and without boundary. Denote its local coordinates by $(U; u_1, \cdots, u_n)$. Consider a C^∞ vector field X which can be expressed locally as $X = \sum_{j=1}^n X^j \frac{\partial}{\partial u_j}$.
   (a) Show that **[10]**
$$i_X(du^1 \wedge \cdots \wedge du^n) = \sum_{j=1}^n (-1)^{j-1} X^j \, du^1 \wedge \cdots \wedge du^{j-1} \wedge du^{j+1} \wedge \cdots du^n.$$
   (b) Suppose Ω is a C^∞ non-vanishing n-form globally defined on M. Denote its local expression in any local chart $(U; u_1, \cdots, u_n)$ by **[15]**
$$\Omega := e^{f_U} \, du^1 \wedge \cdots \wedge du^n.$$
   Consider another n-form ω_U whose local expression in the local chart $(U; u_1, \cdots, u_n)$ is given by:
$$\omega_U := \sum_{j=1}^n \left(\frac{\partial X^j}{\partial u_j} + X^j \frac{\partial f_U}{\partial u_j} \right) \Omega$$
   i. Show that the above local expression is independent of local coordinates.
   ii. Denote $\omega := \omega_U$ on any local chart U. Show that $\int_M \omega = 0$.

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3. (a) Show that for $k \in \mathbb{N} \cup \{0\}$, we have **[5]**
$$H^k_{\text{dR}}(\mathbb{S}^3) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } 3 \\ 0 & \text{otherwise .} \end{cases}$$
   [Remark: You can use results from Example 5.22 if needed.]
   (b) Consider the subsets of $\mathbb{CP}^2 := \{[z_0 : z_1 : z_2] \mid (z_0, z_1, z_2) \in \mathbb{C}^3 \backslash \{(0,0,0)\}\}$: **[15]**
$$U := \mathbb{CP}^2 \backslash \{[1:0:0]\} \text{ and } \Sigma_0 := \{[0 : z_1 : z_2] \in \mathbb{CP}^2 \mid (z_1, z_2) \neq (0,0)\}.$$
   Show that Σ_0 is a deformation retract of U. Please provide the detail, including why Σ_0 is a submanifold of U, and the explicit construction (and verification) the retraction maps Ψ_t.
   (c) Using (a) and (b), find $H^k_{\text{dR}}(\mathbb{CP}^2)$ for all $k \in \mathbb{N} \cup \{0\}$. **Give at least a brief reason for every small step**. You can use the fact that $\mathbb{CP}^2$ is compact. **[15]**

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* End of Paper *

#1

a) Define $h: \mathbb{R}^n \to \mathbb{R}^2$ by $h(x) = (f(x), g(x))$,

then $\Sigma_f \cap \Sigma_g = f^{-1}(0) \cap g^{-1}(0) = h^{-1}((0,0))$.

$\{\nabla f(p), \nabla g(p)\}$ linearly independent

and $f, g \in C^\infty \Rightarrow \exists\, \underset{p \in}{\mathcal{U}} \underset{\mathbb{R}^n}{\subset}$ open set s.t. $\{\nabla f(q), \nabla g(q)\}$ is linearly independent $\forall q \in \mathcal{U}$.

$$[h_{*q}] = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}(q) = \begin{bmatrix} (\nabla f(q))^T \\ (\nabla g(q))^T \end{bmatrix}$$

row reduce

CANNOT BE : $\begin{bmatrix} * & * & \cdots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

$\therefore$ h_{*q} is surjective $\forall q \in \mathcal{U}$

$\Rightarrow$ h is a submersion on $\mathcal{U}$ $\qquad \boxed{h: \mathbb{R}^n \to \mathbb{R}^2}$

$\therefore$ $\underbrace{h^{-1}(0,0) \cap \mathcal{U}}$ is a smooth manifold of dimension $\underset{\dim \mathbb{R}^n}{\overset{\dim \mathbb{R}^2}{n-2}}$

$= \Sigma_f \cap \Sigma_g \cap \mathcal{U}$

(b) From a), $\Sigma_f \cap \Sigma_g \cap \mathcal{U}$ is a C^∞ submanifold of $\mathbb{R}^n$.

$\forall q \in \mathcal{U}$, $\{\nabla f(q), \nabla g(q)\}$ linearly independent

$\Rightarrow \nabla f(q) \neq 0 \Rightarrow f$ is a submersion on $\mathcal{U}$.

$\therefore$ $\Sigma_f \cap \mathcal{U} = f^{-1}(0) \cap \mathcal{U}$ is a C^∞-submanifold of $\mathbb{R}^n$.

$\Sigma_f \cap \mathcal{U}$ is open subset of $\Sigma_f \Rightarrow$ hence $\Sigma_f \cap \mathcal{U}$ is an submfd of Σ_f.

Consider the inclusion maps (which all commute)

$$\begin{tikzcd} \Sigma_f \cap \Sigma_g \cap \mathcal{U} \arrow[r, "i"] \arrow[dr, "j_1"'] & \Sigma_f \cap \mathcal{U} \arrow[r, "j_3"] \arrow[d, "j_2"] & \Sigma_f \\ & \mathbb{R}^n & \end{tikzcd}$$

From previous discussion, $(j_1)_{*q}$, $(j_2)_{*q}$, $(j_3)_{*q}$ are known to be injective $\forall q \in \mathcal{U}$.

$$j_1 = j_2 \circ i \implies (j_1)_* = (j_2)_* \circ i_*$$

$(j_1)_{*q}$ injective $\implies$ i_{*q} injective too.

$$\Rightarrow (j_3 \circ i)_{*q} = \underbrace{(j_3)_{*q} \circ i_{*q}}_{\text{injective}} \text{ is also injective } \forall q \in \mathcal{U}.$$

$\therefore$ $\Sigma_f \cap \Sigma_g \cap \mathcal{U}$ is a submanifold of Σ_f.

#2

(a) $i_X(du^1\wedge\cdots\wedge du^n)$ is an $(n-1)$-form on M^n.

$\therefore$ Its local expression is completely determined by its action on

$$\left(\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_{j-1}},\frac{\partial}{\partial u_{j+1}},\ldots,\frac{\partial}{\partial u_n}\right) \qquad (j=1,2,\ldots,n).$$

$$\left(i_X\left(du^1\wedge\cdots\wedge du^n\right)\right)\left(\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_{j-1}},\frac{\partial}{\partial u_{j+1}},\ldots,\frac{\partial}{\partial u_n}\right)$$

$$=\left(du^1\wedge\cdots\wedge du^n\right)\left(X,\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_{j-1}},\frac{\partial}{\partial u_{j+1}},\ldots,\frac{\partial}{\partial u_n}\right)$$

$$=\left(du^1\wedge\cdots\wedge du^n\right)\left(\sum_{i=1}^n X^i\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_{j-1}},\frac{\partial}{\partial u_{j+1}},\ldots,\frac{\partial}{\partial u_n}\right)$$

For each $i=1,2,\ldots,n$,

$$(du^1\wedge\cdots\wedge du^n)\left(\frac{\partial}{\partial u_i},\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_{j-1}},\frac{\partial}{\partial u_{j+1}},\ldots,\frac{\partial}{\partial u_n}\right)\neq 0$$

<u>ONLY</u> when $i=j$.

$$\therefore \left(i_X(du^1\wedge\cdots\wedge du^n)\right)\left(\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_{j-1}},\frac{\partial}{\partial u_{j+1}},\ldots,\frac{\partial}{\partial u_n}\right)$$

$$=\left(du^1\wedge\cdots\wedge du^n\right)\left(X^j\frac{\partial}{\partial u_j},\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_{j-1}},\frac{\partial}{\partial u_{j+1}},\ldots,\frac{\partial}{\partial u_n}\right)$$

$$=X^j\left((-1)^{j-1}du^j\wedge du^1\wedge\cdots\wedge du^{j-1}\wedge du^{j+1}\wedge\cdots\wedge du^n\right)\left(\frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_{j-1}},\frac{\partial}{\partial u_{j+1}},\ldots,\frac{\partial}{\partial u_n}\right)$$

$$=(-1)^{j-1}X^j$$

$\therefore \quad i_X(du^1\wedge\cdots\wedge du^n)$

$$= \sum_{j=1}^n (-1)^{j-1} X^j \underbrace{du^1\wedge\cdots\wedge du^{j-1}\wedge du^{j+1}\wedge\cdots\wedge du^n}_{\substack{\text{dual elements}\\ \text{of } \frac{\partial}{\partial u_1},\ldots,\frac{\partial}{\partial u_{j-1}},\frac{\partial}{\partial u_{j+1}},\ldots,\frac{\partial}{\partial u_n}.}} \quad ⊠$$

(b) We will check that $d(i_X\Omega) = \omega_u$ on U.

(i) Since $i_X\Omega$ ($\leftarrow$ global, $\leftarrow$ global) is globally defined (given) and d is independent of local coordinates, this will show ω_u is indep. of local coordinates too.

$$d(i_X\Omega) = d\left(i_X\left(e^{f_u}du^1\wedge\cdots\wedge du^n\right)\right)$$

$$= d\left(e^{f_u}\, i_X(du^1\wedge\cdots\wedge du^n)\right)$$

$$= d\left(e^{f_u}\sum_{j=1}^n (-1)^{j-1}X^j du^1\wedge\cdots\wedge du^{j-1}\wedge du^{j+1}\wedge\cdots\wedge du^n\right)$$

$$= \sum_{j=1}^n\left(\sum_{i=1}^n \frac{\partial}{\partial u_i}\left((-1)^{j-1}e^{f_u}X^j\right)\underbrace{du^i\wedge du^1\wedge\cdots\wedge du^{j-1}\wedge du^{j+1}\wedge\cdots\wedge du^n}_{\neq 0 \text{ only when } i=j}\right)$$

$$= \sum_{j=1}^n\left(\frac{\partial}{\partial u_j}\left((-1)^{j-1}e^{f_u}X^j\right)\underbrace{du^j\wedge du^1\wedge\cdots\wedge du^{j-1}\wedge du^{j+1}\wedge\cdots\wedge du^n}_{=(-1)^{j-1}du^1\wedge\cdots\wedge du^n}\right)$$

$$= \sum_{j=1}^{n} \Big(\underbrace{(-1)^{2j-2}}_{1} \left(e^{f_u} \frac{\partial f_u}{\partial u_j} X^j + e^{f_u} \frac{\partial X^j}{\partial u_j} \right) du^1 \wedge \cdots \wedge du^n \Big)$$

$$= \sum_{j=1}^{n} \left(\frac{\partial X^j}{\partial u_j} + X^j \frac{\partial f_u}{\partial u_j} \right) \Omega$$

$$\Omega = e^{f_u} du^1 \wedge \cdots \wedge du^n .$$

$$= \omega_u .$$

$\square$

(ii) M is compact without boundary.

Stokes' Theorem

$$\Longrightarrow \int_M \omega = \int_M \underbrace{d(i_X \Omega)}_{\text{exact}} = \int_{\partial M} i_X \Omega = 0 .$$

$\partial M = \emptyset$

$\blacksquare$

#3

(a) $S^3 = \{\vec{x} \in \mathbb{R}^4 : |\vec{x}| = 1\}$ is compact, orientable, without boundary

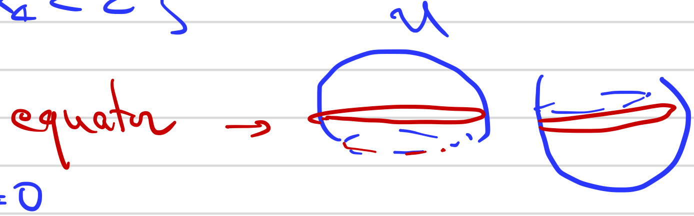
$$\Longrightarrow H^3(S^3) = \mathbb{R}$$

$\uparrow$ dim $= 3$.

let $\mathcal{U} = \{\vec{x} \in S^3 \subset \mathbb{R}^4 : x_4 > -\varepsilon\}$ where $\varepsilon \ll 1$.

$V = \{\vec{x} \in S^3 \subset \mathbb{R}^4 : x_4 < \varepsilon\}$

Each $\mathcal{U}$ and V are contractible $\Rightarrow H^k(\mathcal{U}) = H^k(V) = 0$ $\forall k \geq 1$.



$\mathcal{U} \cup V = S^3$

and $\mathcal{U} \cap V$ deformation retract onto the equator $\{\vec{x} \in S^3 : x_4 = 0\} \cong S^2$. $\Rightarrow H^k(\mathcal{U} \cap V) = H^k(S^2)$ $\forall k \geq 0$.

Consider the Mayer-Vietoris sequence:

dim:

$$0 \to \underset{1}{H^0(S^3)} \to \underset{(1+1)}{H^0(\mathcal{U}) \oplus H^0(V)} \to \underset{1}{H^0(S^2)} \to$$

connected

$$\to \underset{x}{H^1(S^3)} \to \underset{0}{H^1(\mathcal{U}) \oplus H^1(V)} \to \underset{0}{H^1(S^2)} \to$$

from notes.

$$\to \underset{y}{H^2(S^3)} \to \underset{0}{H^2(\mathcal{U}) \oplus H^2(V)}$$

Consider alternating sums (of short exact sequences with 0 ends)

$0 = 1 - (1+1) + 1 - x$

$0 = y$

$\Rightarrow \quad x = 0, \ y = 0$

$\therefore$

$H^0(S^3) = \mathbb{R}$ $\qquad$ $H^k(S^3) = 0$

$H^1(S^3) = 0$ $\qquad$ $\forall k > 3$

$H^2(S^3) = 0$ $\qquad$ " dim.

$H^3(S^3) = 1$

$\boxtimes$

(b) $\Sigma_0 \subset \mathcal{U}$ because $[0 : z_1 : z_2] \neq [1:0:0]$ $\forall (z_1, z_2) \neq (0,0)$

Parametrize Σ_0 by

$F_1(z) := [0 : 1 : z] : \mathbb{C} \to \Sigma_0$

$F_2(w) := [0 : w : 1] : \mathbb{C} \to \Sigma_0$

then clearly $F_1^{-1} \circ F_2$ and $F_2^{-1} \circ F_1$ are maps $z \mapsto \frac{1}{z}$, hence holomorphic on the overlap $\mathbb{C} \setminus \{0\}$.

Parametriz $\mathcal{U} \leftarrow$ open set of $\mathbb{CP}^2$ by standard parametizations of $\mathbb{CP}^2$

e.g. $G_0(z_1, z_2) : \mathbb{C}^2 \to \mathbb{CP}^2$, $\quad G_1 : \mathbb{C}^2 \to \mathbb{CP}^2$, $G_2 : \mathbb{C}^2 \to \mathbb{CP}^2$

$\quad$ " $\qquad\qquad\qquad\qquad$ similarly.

$[1 : z_1 : z_2]$

Then, $G_1^{-1} \circ \iota \circ F_1$ is not defined.

$G_2^{-1} \circ \iota \circ F_1(z)$
$= G_2^{-1}([0:1:z])$
$= (0, z)$

$G_3^{-1} \circ \iota \circ F_1(z)$
$= G_3^{-1}([0:\frac{1}{z}:1])$
$= (0, \frac{1}{z})$

are all C^∞ on its domain.

$$[D(G_2^{-1} \circ \iota \circ F_1)] = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

$$[D(G_3^{-1} \circ \iota \circ F_1)] = \begin{bmatrix} 0 \\ -\frac{1}{z^2} \end{bmatrix} \neq 0$$

The other combinations $G_j^{-1} \circ \iota \circ F_2$ are similar.

$\therefore$ ι is an immersion $\Rightarrow$ Σ_0 is a submfd of $\mathcal{U}$.

Next we verify that Σ_0 is a deformation retract of $\mathcal{U}$:

Define $\psi_t : \mathcal{U} \times [0,1] \to \mathcal{U}$ by

$$\psi_t([z_0:z_1:z_2]) = [(1-t)z_0 : z_1 : z_2]$$

(0) First argue $\mathrm{Image}(\psi_t) \subset \mathcal{U}$ $\forall t \in [0,1]$:

Suppose not, $\exists t_0 \in [0,1]$, $[z_0:z_1:z_2] \in \mathcal{U}$ s.t.

$$\psi_t([z_0:z_1:z_2]) = [1:0:0]$$

$$\Rightarrow [(1-t_0)z_0 : z_1 : z_2] = [1:0:0]$$

$$\Rightarrow \begin{cases} z_1 = 0 \\ z_2 = 0 \end{cases} \text{ and } \underbrace{(1-t_0)z_0 \neq 0}_{\Downarrow\ z_0 \neq 0} \Rightarrow \begin{array}{l} [z_0:z_1:z_2] \\ = [1:0:0] \\ \notin \mathcal{U}. \end{array}$$

※

① $\psi_0([z_0:z_1:z_2]) = [(1-0)z_0 : z_1 : z_2]$
$\qquad\qquad\qquad\quad = [z_0 : z_1 : z_2]$

$\Rightarrow \psi_0 = \mathrm{id}_{\mathcal{U}}$.

② $\psi_1([z_0:z_1:z_2]) = [0:z_1:z_2] \in \Sigma_0$.

③ If $[z_0:z_1:z_2] \in \Sigma_0$, then

$z_0 = 0$ and so

$$\psi_t([\underset{\overset{\shortparallel}{0}}{z_0}:z_1:z_2]) = \psi_t([0:z_1:z_2])$$
$$= [(1-t)0 : z_1 : z_2]$$
$$= [0:z_1:z_2]$$

$\Rightarrow \psi_t\big|_{\Sigma_0} = \mathrm{id}_{\Sigma_0} \quad \forall t \in [0,1]$.

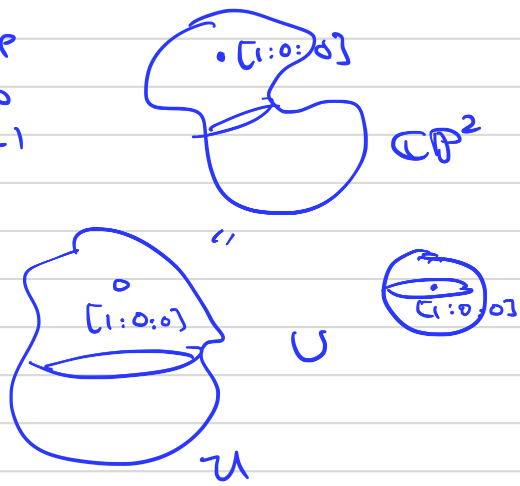
$\therefore$ ψ_t defines a deformation retract of $\mathcal{U}$ onto Σ_0.

(c) Let $\mathcal{U}$ as in (b)

$V = B_\varepsilon([1:0:0]) \leftarrow$ contractible $\therefore H^k(V)=0 \ \forall k \geq 1$

$= 4$-dim'l ball centered at $[1:0:0]$. with small radius ε.

(Note $\dim_{\mathbb{R}} \mathbb{CP}^2 = \underline{4}$)



then $\mathcal{U} \cup V = \mathbb{CP}^2$,

$\mathcal{U} \cap V = B_\varepsilon([1:0:0]) \setminus \{[1:0:0]\}$

deformation retract onto $S^3 = \partial B^4$.

$$H^k(\mathcal{U}) \underset{\substack{\uparrow \\ \text{from} \\ \text{deform.} \\ \text{retract}}}{=} H^k(\Sigma_0) \underset{\substack{\uparrow \\ \text{diffeo} \\ \text{invariant}}}{=} H^k(\mathbb{CP}^1) = H^k(S^1). \quad \forall k \geq 0.$$

Consider the Mayer-Vietoris sequence:

$$0 \to \underbrace{H^0(\mathbb{CP}^2)}_{1} \to \underbrace{H^0(S^2) \oplus H^0(B^4)}_{(1+1)} \to \underbrace{H^0(S^3)}_{1} \to$$

dim: connected, connected, connected

$$\to \underbrace{H^1(\mathbb{CP}^2)}_{x} \to \underbrace{H^1(S^2) \oplus H^1(B^4)}_{(0\ +\ 0)} \to \underbrace{H^1(S^3)}_{0} \to$$

from notes, •, from (a)

$$\to \underbrace{H^2(\mathbb{CP}^2)}_{y} \to \underbrace{H^2(S^2) \oplus H^2(B^4)}_{(1\ +\ 0)} \to \underbrace{H^2(S^3)}_{0} \to$$

from notes, •, from (a)

$$\to \underbrace{H^3(\mathbb{CP}^2)}_{z} \to \underbrace{H^3(S^2) \oplus H^3(B^4)}_{(0\ +0)} \to \underbrace{H^3(S^3)}_{1} \to$$

•, from (a) / notes.

$$\to \underbrace{H^4(\mathbb{CP}^2)}_{1} \to \underbrace{H^4(S^2)}_{0} \oplus \underbrace{H^4(B^4)}_{0} \to \cdots$$

compact, orientable, $\leftarrow$ complex manifold.
4-manifold without boundary

$$\therefore H^4(\mathbb{CP}^2) = \mathbb{R}$$

Consider alternating sum:

$$\begin{cases} 0 = 1-2+1-x \\ 0 = y-1 \\ 0 = z \end{cases} \Rightarrow \begin{matrix} x = 0 \\ y = 1 \\ z = 0 \end{matrix}$$

$$\therefore H^k(\mathbb{CP}^2) = \begin{cases} \mathbb{R} & \text{if } k=0 \\ 0 & \text{if } k=1 \\ \mathbb{R} & \text{if } k=2 \\ 0 & \text{if } k=3 \\ \mathbb{R} & \text{if } k=4 \\ 0 & \text{if } k\geq 4 \end{cases}$$

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