香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

FINAL EXAMINATION

Course Code:	MATH 1024
Course Title:	Honors Calculus II
Semester:	Spring 2020-21
Date and Time:	12:30PM - 3:30PM, 15 May 2021

Instructions

- Absolutely **NO DISCUSSION**, online or offline, with any person other than the instructor.
- Posting anything on social media or any public internet website is **NOT** allowed.
- It is an **OPEN-NOTES** and **OPEN-INTERNET** exam. You can do searching on the internet (including using WolframAlpha), but no posting and discussion.
- Only results discussed in lectures and tutorials, and results proved in homework can be directly quoted.
- You must **SHOW YOUR WORK** and **JUSTIFY YOUR STEPS** to receive credits in every problem in Part B.
- Some problems in Part B are structured into several parts. You can quote the results stated in the preceding parts to do the next part, even if you cannot do the preceding parts.
- One page 1 of your work, write the following statement and sign:

  > "I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

  Your signature

- You need to **SIGN** on the top right corner of **EVERY PAGE** of your work.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

Part A - Short Questions (30 points)

Recommended timing: the whole Part A $<$ 35min.
Instruction: Write down your answers or solutions in your own answer sheets. **State the question and part numbers clearly**.

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1. Name one mathematician who was born in April. **[1]**

2. Given that F is C^1 on $\mathbb{R}$ and $F'(x) = f(x)$ for any $x \in \mathbb{R}$, which of the following must be true? List ALL correct answer(s). **[5]**

   A. $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$

   B. $\displaystyle\int_a^b F(x)\,dx = f(b) - f(a)$

   C. $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$

   D. $\displaystyle\int_a^b F'(x)\,dx = f(b) - f(a)$

   E. $\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$

3. Suppose f is continuous on $\mathbb{R}$. For any $x \in \mathbb{R}$, we let $F(x) := \displaystyle\int_0^x f(t)\,dt$. Which of the following must be true? List ALL correct answer(s). **[4]**

   A. $F(x) \geq 0 \ \ \forall x \geq 0$.

   B. $\displaystyle\lim_{x\to a} F(x)$ exists for any $a \in \mathbb{R}$.

   C. F is continuous on $\mathbb{R}$

   D. F is differentiable on $\mathbb{R}$.

4. Given that $\{P_n(x)\}_{n=0}^{2021}$ is sequence of Riemann integrable functions on $[-1,1]$ such that for any m, n we have **[2]**

$$\int_{-1}^1 P_m(x)P_n(x)\,dx = \begin{cases} \frac{2}{2n+1} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

   Given that $f(x) = \sum_{n=0}^{2021} a_n P_n(x)$ where $a_n \in \mathbb{R}$ are constants, which ONE of the following true? Write down the correct answer. No need to explain.

   A. $a_n = \displaystyle\int_{-1}^1 f(x)P_n(x)\,dx$

   B. $a_n = \dfrac{2}{2n+1}\displaystyle\int_{-1}^1 f(x)P_n(x)\,dx$

   C. $a_n = \dfrac{2n+1}{2}\displaystyle\int_{-1}^1 f(x)P_n(x)\,dx$

   D. $a_n = \dfrac{1}{\pi}\displaystyle\int_{-1}^1 f(x)P_n(x)\,dx$

   E. $a_n = \dfrac{1}{2}\displaystyle\int_{-1}^1 f(x)P_n(x)\,dx$

5. What is the following sum? Write down the correct answer. No need to explain. **[2]**

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}-\frac{1}{2}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}-\frac{1}{4}+\frac{1}{17}+\frac{1}{19}+\frac{1}{21}+\frac{1}{23}-\frac{1}{6}+\cdots$$

   A. 0
   B. $\log 2$
   C. $\log 4$
   D. $\frac{\pi^2}{6}$
   E. $-\frac{1}{12}$

6. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. **[5]**
   In each _____ below, write down one of the symbols: $\Longrightarrow$, $\Longleftarrow$, or $\Longleftrightarrow$.
   (a) $\sum_{n=1}^{\infty} a_n$ converges _____ $\sum_{n=2021}^{\infty} a_n$ converges
   (b) $\sum_{n=1}^{\infty} |a_n|$ converges _____ $\sum_{n=1}^{\infty} a_n$ converges
   (c) $\sum_{n=1}^{\infty} a_n$ converges _____ $\lim\limits_{n\to\infty} a_n = 0$
   (d) $\sum_{n=1}^{\infty} (-1)^n a_n$ converges _____ $\lim\limits_{n\to\infty} a_n = 0$
   (e) $\sum_{n=1}^{\infty} a_n$ diverges _____ $\frac{1}{n} \le a_n \ \forall n \in \mathbb{N}$

7. According to the proofs done in lectures, which of the following is/are consequence(s) of the comparison test (for infinite series)? List ALL correct answer(s). **[5]**
   [Remark: If (B) is used to prove (C), and (B) was proved by (A), then (C) is also a consequence of (A).]
   A. Limit comparison test
   B. Ratio test
   C. Root test
   D. Absolute convergence test
   E. Alternating series test

8. Consider the following parametric "curve" $\mathbf{r}(t) : [-1,1] \to \mathbb{R}^2$: **[6]**

$$\mathbf{r}(t) = \begin{cases} (t,0) & \text{if } t \neq 0 \\ (0,1) & \text{if } t = 0 \end{cases}.$$

   Prove that the curve $\{\mathbf{r}(t)\}_{t\in[-1,1]}$ is rectifiable.

Part B - Long Questions (70 points): Answer ALL THREE problems

Recommended timing:
Instructions: Write your solutions on your own answer sheets. **Clearly indicate the question and part numbers.**

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1. Let $f : [0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ be two functions satisfying the conditions:
   (I) f is C^1 on $[0, \infty)$.
   (II) g is continuous on $[0, \infty)$.
   (III) f is increasing on $[0, \infty)$.
   (IV) $f(0) > 0$ and f is unbounded on $[0, \infty)$.
   (V) $\displaystyle\int_0^{+\infty} g(t)\,dt$ converges.

   Answer the following questions:
   (a) Show that **[6]**
   $$\int_0^x f(t)g(t)\,dt = f(x)\int_0^x g(t)\,dt - \int_0^x \left(\int_0^t g(y)\,dy\right) f'(t)\,dt.$$
   for any $x \geq 0$.
   (b) Show that **[14]**
   $$\lim_{x\to+\infty} \frac{1}{f(x)} \int_0^x f(t)g(t)\,dt = 0.$$

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2. For any $n \in \mathbb{N}$, we let
   $$a_n := \left(\frac{(2n-1)(2n-3)\cdots(3)(1)}{(2n)(2n-2)(2n-4)\cdots(4)(2)}\right)^p = \left(\frac{(2n-1)!!}{(2n)!!}\right)^p,$$
   where p is a constant.
   (a) Suppose (for this part) that $p > 2$, determine whether $\displaystyle\sum_{n=1}^{\infty} a_n$ converges or diverges. Justify your claim. **[12]**
   (b) From now on we suppose $p = 2$.
   i. Show that: **[10]**
   $$\lim_{n\to\infty}\left((n\log n)\frac{a_n}{a_{n+1}} - (n+1)\log(n+1)\right) < 0.$$
   ii. Hence, show that $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2$ diverges. **[5]**

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3. Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function which is C^∞ on $\mathbb{R}$ satisfying
   $$\int_0^{2\pi} f(t)\,dt = 0.$$
   We express f as a complex Fourier series $f(t) = \sum_{n\in\mathbb{Z}} c_n e^{int}$ on $(0, 2\pi)$.

(a) What is c_0 equal to? Explain. **[3]**

(b) For each $m \in \mathbb{N}$, we let **[14]**

$$Q_m(t) := (t-1^2)(t-2^2)\cdots(t-m^2),$$

and denote $a_{m,j}$ to be the coefficients of $Q_m(t)$, i.e.

$$Q_m(t) = \sum_{j=0}^{m} a_{m,j} t^j.$$

Prove that for each $m \in \mathbb{N}$, we have

$$\sum_{j=0}^{m} a_{m,j} \int_0^{2\pi} f^{(j)}(t)^2 \, dt \geq 0,$$

and equality holds if and only if $c_{m+1} = c_{m+2} = \cdots = 0$.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function which is C^2 on $\mathbb{R}$. Show that **[6]**

$$\int_0^{2\pi} g''(t)^2 \, dt - 5\int_0^{2\pi} g'(t)^2 \, dt + 4\int_0^{2\pi} g(t)^2 \, dt \geq \frac{2}{\pi}\left(\int_0^{2\pi} g(t)\, dt\right)^2.$$

- End of Paper -