Part A

#1. Gauss / Euler / Poincaré
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#2. A, E

#3. B, C, D

#4. C

#5. C

#6. (a) $\Leftarrow$
(b) $\Rightarrow$
(c) $\Rightarrow$
(d) $\Rightarrow$
(e) $\Leftarrow$

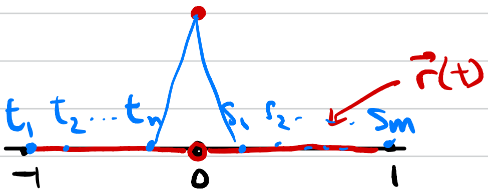
#7. A, B, C, D

#8. For any partition P of $[-1,1]$, we let $P' = P \cup \{0\}$.
Write

$$P' = \{ \underset{-1}{\underset{\shortparallel}{t_0}} < t_1 < \cdots < t_n < 0 < s_1 < \cdots < \underset{1}{\underset{\shortparallel}{s_m}} \}$$

Then

$$\begin{aligned}
\ell_P \le \ell_{P'} &= |\vec{r}(t_1) - \vec{r}(t_0)| + \cdots + |\vec{r}(t_n) - \vec{r}(t_{n-1})| \\
&\quad + |\vec{r}(0) - \vec{r}(t_n)| + |\vec{r}(s_1) - \vec{r}(0)| \\
&\quad + |\vec{r}(s_2) - \vec{r}(s_1)| + \cdots + |\vec{r}(s_m) - \vec{r}(s_{m-1})| \\
&= (t_1 - t_0) + (t_2 - t_1) + \cdots + (t_n - t_{n-1}) \\
&\quad + \overset{\le 1+|t_n|}{\sqrt{t_n^2+1}} + \sqrt{1+s_1^2} \le 1+s_1 \\
&\quad + (s_2 - s_1) + (s_3 - s_2) + \cdots + (s_m - s_{m-1}) \\
&\le t_n - t_0 + 1 + |t_n| + 1 + s_1 + s_m - s_1 \\
&= t_n - (-1) + 1 - t_n + 1 + s_1 + 1 - s_1 \\
&= 4
\end{aligned}$$



$\therefore \sup_P \ell_P \le 4 \Rightarrow \vec{r}(t)$ is rectifiable.

Part B

#1 (a) Denote $G(x) := \int_0^x g(t)\,dt \overset{g \text{ continuous}}{\Longrightarrow} G'(x) = g(x)$ on $[0,\infty)$

By integration-by-parts, we have:

$$\int_0^x f(t)\,g(t)\,dt = \int_0^x f(t)\,d(G(t))$$

$$= \Big[f(t)\,G(t)\Big]_{t=0}^{t=x} - \int_0^x G(t)\,d(f(t))$$

$$= f(x)G(x) - \cancel{f(0)\underbrace{G(0)}_{\int_0^0 g(t)dt=0}} - \int_0^x G(t)\,f'(t)\,dt$$

$$= f(x)\int_0^x g(t)\,dt - \int_0^x \Big(\underbrace{\int_0^t g(y)\,dy}_{G(t)}\Big) f'(t)\,dt. \qquad \square$$

(b) Let $\int_0^{+\infty} g(t)\,dt = L$, then $\lim_{x\to+\infty} G(x) = \lim_{x\to+\infty}\int_0^x g(t)\,dt = L$.

Write $\underbrace{\int_0^x g(t)\,dt}_{G(x)} = L + h(x)$ where $h(x) \in o(1)$ as $x \to +\infty$.

Using (a), we have

$$\frac{1}{f(x)}\int_0^x f(t)g(t)\,dt = \frac{1}{f(x)}\left(f(x)\,G(x) - \int_0^x G(t)\,f'(t)\,dt\right)$$

$$= G(x) - \frac{1}{f(x)}\int_0^x G(t)\,f'(t)\,dt$$

$$= L + h(x) - \frac{1}{f(x)}\int_0^x \big(L + h(t)\big)\,f'(t)\,dt$$

$$= L + h(x) - \frac{1}{f(x)}\Big(L\int_0^x \underbrace{f'(t)}_{\text{continuous}}\,dt + \int_0^x h(t)\,f'(t)\,dt\Big)$$

$$= L + h(x) - \frac{1}{f(x)}\Big(L\,(f(x) - f(0)) + \int_0^x h(t)\,f'(t)\,dt\Big)$$

$$= h(x) + \frac{Lf(0)}{f(x)} - \frac{1}{f(x)}\int_0^x h(t)\,f'(t)\,dt. \qquad \text{—— } (*)$$

As f is unbounded and increasing on $[0, +\infty)$,

we have $\lim_{x \to +\infty} f(x) = +\infty$.

Recall that $h(x) \to 0$ as $x \to +\infty$,

we have

$$h(x) + \frac{Lf(0)}{f(x)} \to 0 \quad \text{as} \quad x \to +\infty.$$

$\therefore$ It suffices to show $\frac{1}{f(x)} \int_0^x h(t) f'(t)\, dt \to 0$ as $x \to +\infty$.

Note that $h(x) \to 0$ as $x \to +\infty$.

$\forall \varepsilon > 0$, $\exists K > 0$ s.t. $|h(x)| < \varepsilon/2 \quad \forall x > K$.

Then for any $x > K$:

$$\left| \frac{1}{f(x)} \int_0^x h(t) f'(t)\, dt \right| \le \frac{1}{f(x)} \int_0^x |h(t) f'(t)|\, dt$$

$f(x) \ge f(0) > 0$

$$\le \frac{1}{f(x)} \left(\int_0^K |h(t)| \underbrace{f'(t)}_{} dt + \int_K^x |h(t)| f'(t)\, dt \right)$$

$f \nearrow$ so $f' \ge 0$

$$\le \frac{1}{f(x)} \left(\max_{[0,K]} |h| \int_0^K f'(t)\, dt + \frac{\varepsilon}{2} \int_K^x f'(t)\, dt \right)$$

$$= \frac{1}{f(x)} \left(\max_{[0,K]} |h| \left(f(K) - f(0)\right) + \frac{\varepsilon}{2} \left(f(x) - \underbrace{f(K)}_{+}\right) \right)$$

$$< \frac{\max_{[0,K]} |h| \left(f(K) - f(0)\right)}{f(x)} + \frac{\varepsilon}{2}$$

as $f(K) \ge f(0) > 0$.

Since $f(x) \to +\infty$, $\exists N > 0$ s.t. $f(x) > \frac{2}{\varepsilon} \max_{[0,K]} |h| \cdot (f(K) - f(0))$

$\forall x > N$.

Then for any $x > \max(N, K)$, we have

$$\left| \frac{1}{f(x)} \int_0^x h(t) f'(t)\, dt \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In other words, we have $\lim_{x \to +\infty} \frac{1}{f(x)} \int_0^x h(t) f'(t)\, dt = 0$.

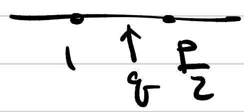
Combine with $(*)$, we proved $\lim_{x \to +\infty} \frac{1}{f(x)} \int_0^x g(t) f(t)\, dt = 0$.

#2. (a) $$\frac{a_{n+1}}{a_n} = \left(\frac{(2n+1)!!}{(2n+2)!!}\cdot\frac{(2n)!!}{(2n-1)!!}\right)^p = \left(\frac{2n+1}{2n+2}\right)^p$$

$$= \left(1-\frac{1}{2n+2}\right)^p = 1-\frac{p}{2n+2}+o\left(\frac{1}{2n+2}\right) \text{ as } n\to\infty.$$

Consider $b_n = \frac{1}{n^q}$, then $\frac{b_{n+1}}{b_n} = \left(1-\frac{1}{n+1}\right)^q = 1-\frac{q}{n+1}+o\left(\frac{1}{n+1}\right)$

When $p>2$, pick $q\in(1,\frac{p}{2})$



then $\frac{p}{2n+2} > \frac{q}{(n+1)}$

$\Rightarrow -\frac{p}{2n+2} < -\frac{q}{n+1}$

$\Rightarrow 1-\frac{p}{2n+2}+o\left(\frac{1}{2n+2}\right) < 1-\frac{q}{n+1}+o\left(\frac{1}{n+1}\right) \quad \forall n>>1.$

$\therefore \frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n} \quad \forall n>>1$

As $\sum b_n = \sum \frac{1}{n^q}$ converges with $q>1$.

$\therefore$ By ratio test $\sum a_n$ converges.

(b) i. $p=2$

$$\lim_{n\to\infty}\left(n\log n\cdot\frac{a_n}{a_{n+1}} - (n+1)\log(n+1)\right)$$

$$= \lim_{n\to\infty}\left(n\log n\cdot\left(\frac{2n+2}{2n+1}\right)^2 - (n+1)\log(n+1)\right)$$

$$= \lim_{n\to\infty}\left(n\log n\cdot\left(1+\frac{1}{2n+1}\right)^2 - (n+1)\log(n+1)\right)$$

$$= \lim_{n\to\infty}\left(n\log n\cdot\left(1+\frac{2}{2n+1}+\frac{1}{(2n+1)^2}\right) - (n+1)\log(n+1)\right)$$

$$= \lim_{n\to\infty}\left(n\log n+\frac{2n}{2n+1}\log n+\frac{n}{(2n+1)^2}\log n - (n+1)\left(\log n+\log\left(1+\frac{1}{n}\right)\right)\right)$$

$$= \lim_{n\to\infty}\left(\left(\frac{2n}{2n+1}-1\right)\log n + \frac{n}{(2n+1)^2}\log n - (n+1)\log\left(1+\frac{1}{n}\right)\right)$$

$$= \lim_{n\to\infty}\left(-\frac{1}{2n+1}\log n - \log\left(1+\frac{1}{n}\right)^n - \log\left(1+\frac{1}{n}\right) - \frac{n}{(2n+1)^2}\log n\right)$$

Note that $\log n\in o(n)$ so $\frac{\log n}{2n+1}\to 0$ and $\frac{n}{(2n+1)^2}\log n\to 0$.

$= 0-\log e - \log(1+0) - 0 = -1 < 0$ ☒

ii. From (i), we know

$$(n\log n)\frac{a_n}{a_{n+1}} - (n+1)\log(n+1) < 0 \quad \forall n >> 1.$$

$$\Rightarrow \frac{n\log n}{(n+1)\log(n+1)} < \frac{a_{n+1}}{a_n}$$

$$\Rightarrow \frac{\frac{1}{(n+1)\log(n+1)}}{\frac{1}{n\log n}} < \frac{a_{n+1}}{a_n} \quad \forall n >> 1$$

Note that $\sum \frac{1}{n\log n}$ diverges by integral test:

$$\int_1^{+\infty} \underbrace{\frac{1}{x\log x}}_{\text{decreasing}} dx = \int_1^{+\infty} \frac{d(\log x)}{\log x} = \left[\log(\log(x))\right]_{x=1}^{x\to+\infty} = +\infty.$$

$\therefore$ By ratio test, $\sum a_n$ diverges too.

#3

(a) $0 = \int_0^{2\pi} f(t)\,dt = \int_0^{2\pi} c_0 + \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n e^{int}\,dt = 2\pi c_0 + \underbrace{\sum_{n\in\mathbb{Z}\setminus\{0\}} \int_0^{2\pi} c_n e^{int}\,dt}_{0}$

Since $\int_0^{2\pi} e^{int}\,dt = \left[\frac{1}{in} e^{int}\right]_0^{2\pi} = \frac{1}{in}(1-1) = 0 \quad \forall n \in \mathbb{Z}\setminus\{0\}$.

$\therefore \boxed{c_0 = 0}$

(b) $f(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n e^{int}$ (from (a), $c_0 = 0$)

$\Rightarrow f'(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n in e^{int}$

$\Rightarrow f''(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n (in)^2 e^{int}$

Inductively, $f^{(j)}(t) = \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n (in)^j e^{int}$

As $\langle e^{int}, e^{imt}\rangle = \int_0^{2\pi} e^{int} e^{-imt}\,dt = \begin{cases} 0 & \text{if } m \neq n \\ 2\pi & \text{if } m = n \end{cases}$,

we have $\int_0^{2\pi} f^{(j)}(t)^2\,dt = \int_0^{2\pi} f^{(j)}(t)\,\overline{f^{(j)}(t)}\,dt = \langle f^{(j)}, f^{(j)}\rangle$

as $f^{(j)}(t) \in \mathbb{R}$

$= \left\langle \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n (in)^j e^{int}, \sum_{m\in\mathbb{Z}\setminus\{0\}} c_m (im)^j e^{imt} \right\rangle$

$= \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n \bar{c}_n (in)^j \overline{(in)^j}\, \|e^{int}\|^2 = \sum_{n\in\mathbb{Z}\setminus\{0\}} |c_n|^2 |n|^{2j} \cdot 2\pi$

$\sum_{j=0}^{m} a_{m,j} \int_0^{2\pi} f^{(j)}(t)^2\,dt$

$= \sum_{j=0}^{m} a_{m,j} \sum_{n\in\mathbb{Z}\setminus\{0\}} |c_n|^2 |n|^{2j}\, 2\pi$

$= \sum_{n\in\mathbb{Z}\setminus\{0\}} |c_n|^2 \underbrace{\sum_{j=0}^{m} a_{m,j} |n|^{2j}}_{= Q_m(|n|^2)} \cdot 2\pi$

$= \sum_{n\in\mathbb{Z}\setminus\{0\}} |c_n|^2 (|n|^2 - 1^2)(|n|^2 - 2^2)\cdots(|n|^2 - m^2) \cdot 2\pi$

Note that when $|n| \le m$, we have

$$(|n|^2-1^2)\cdots(|n|^2-m^2) = (|n|^2-1^2)\cdots\underbrace{(|n|^2-|n|^2)}_{=0}\cdots(|n|^2-m^2) = 0$$

$$\therefore \sum_{j=0}^{m} a_{m,j} \int_0^{2\pi} f^{(j)}(t)^2\,dt = \sum_{|n|>m} \underbrace{|c_n|^2}_{\ge 0} \underbrace{(|n|^2-1)(|n|^2-2^2)\cdots(|n|^2-m^2)}_{>0 \text{ as } |n|>m} \cdot 2\pi$$

$$\ge 0.$$

Equality holds iff $c_n = 0$ whenever $|n| > m$,

Since $f(t) \in \mathbb{R}$, we have $\overline{f(t)} = f(t)$

$$\underbrace{\sum_{n=-\infty}^{-1} \bar{c}_n e^{-int} + \sum_{n=1}^{\infty} \bar{c}_n e^{-int}}_{\parallel} = \sum_{n=1}^{\infty} c_n e^{int} + \sum_{n=-\infty}^{-1} c_n e^{int}$$

$$\sum_{n=1}^{\infty} \overline{c_{-n}} e^{int} + \sum_{n=-\infty}^{-1} \overline{c_{-n}} e^{int}$$

$$\therefore \quad c_n = \overline{c_{-n}} \quad \forall n \in \mathbb{Z}\setminus\{0\}.$$

$\therefore$ $c_n = 0$ whenever $|n| > m$

$\Leftrightarrow$ $c_n = 0$ whenever $n > m$.

$\therefore$ Equality holds iff $c_{m+1} = c_{m+2} = \cdots = 0$.

(b) Take $m=2$, $Q_2(t) = (t-1^2)(t-2^2) = t^2 - 5t + 4$. $\rightarrow$ $\begin{cases} a_{2,0} = 4 \\ a_{2,1} = -5 \\ a_{2,2} = 1 \end{cases}$.

Let $f(t) = g(t) - \underbrace{\frac{1}{2\pi}\int_0^{2\pi} g(y)\,dy}_{\text{constant}}$

then f is 2π-periodic with

$$\int_0^{2\pi} f(t)\,dt = \int_0^{2\pi} g(t)\,dt - \int_0^{2\pi} \underbrace{\left(\frac{1}{2\pi}\int_0^{2\pi} g(y)\,dy\right)}_{\text{constant}} dt$$

$$= \int_0^{2\pi} g(t)\,dt - \cancel{2\pi}\cdot\frac{1}{\cancel{2\pi}}\int_0^{2\pi} g(y)\,dy$$

$$= 0.$$

From (a), we have

$$0 \le \int_0^{2\pi} f''(t)^2\,dt - 5\int_0^{2\pi} f'(t)^2\,dt + 4\int_0^{2\pi} f(t)^2\,dt \quad \text{— } (*)$$

Note $f' = g'$ and $f'' = g''$, and

constants

$$\int_0^{2\pi} f(t)^2\,dt = \int_0^{2\pi} \left(g(t) - \frac{1}{2\pi}\int_0^{2\pi} g(y)\,dy\right)^2 dt = \int_0^{2\pi} g(t)^2 - \frac{2g(t)}{2\pi}\int_0^{2\pi} g(y)\,dy + \left(\frac{1}{2\pi}\int_0^{2\pi} g(y)\,dy\right)^2 dt$$

$$= \int_0^{2\pi} g(t)^2\,dt - \frac{1}{\pi}\int_0^{2\pi} g(t)\,dt \cdot \int_0^{2\pi} g(y)\,dy + 2\pi\left(\frac{1}{2\pi}\int_0^{2\pi} g(y)\,dy\right)^2$$

$$= \int_0^{2\pi} g(t)^2\,dt - \frac{1}{\pi}\left(\int_0^{2\pi} g(t)\,dt\right)^2 + \frac{1}{2\pi}\left(\int_0^{2\pi} g(t)\,dt\right)^2$$

$$= \int_0^{2\pi} g(t)^2\,dt - \frac{1}{2\pi}\left(\int_0^{2\pi} g(t)\,dt\right)^2 .$$

From $(*)$, we have

$$0 \le \int_0^{2\pi} (g''(t))^2\,dt - 5\int_0^{2\pi} (g'(t))^2\,dt + 4\left(\int_0^{2\pi} g(t)^2\,dt - \frac{1}{2\pi}\left(\int_0^{2\pi} g(t)\,dt\right)^2\right)$$

$$= \int_0^{2\pi} (g''(t))^2\,dt - 5\int_0^{2\pi} (g'(t))^2\,dt + 4\int_0^{2\pi} g(t)^2\,dt - \frac{2}{\pi}\left(\int_0^{2\pi} g(t)\,dt\right)^2$$

yields the desired result.