Parseval's identity

$$\underbrace{f(x)} = c\cdot\underbrace{1} + \sum_{n=1}^{\infty} a_n \underbrace{\cos nx} + \sum_{n=1}^{\infty} b_n \underbrace{\sin nx}.$$

period 2π

$$\langle g, h\rangle := \int_0^{2\pi} g(x)\,\overline{h(x)}\,dx$$

If $\langle v_i, v_j\rangle = 0 \ \forall i\neq j$

then $\langle \sum_i c_i \vec{v}_i, \sum_j c_j \vec{v}_j\rangle$

$= \sum_{i,j} c_i c_j \langle \vec{v}_i, \vec{v}_j\rangle = \sum_i c_i^2 |\vec{v}_i|^2$

$$\int_0^{2\pi} f(x)^2\,dx = \langle f, f\rangle$$

$$= \left\langle c\cdot 1 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx,\ c\cdot 1 + \sum_{m=1}^{\infty} a_m \cos mx + \sum_{m=1}^{\infty} b_m \sin mx\right\rangle$$

$$= c^2\|1\|^2 + \sum_{n=1}^{\infty} a_n^2 \|\cos nx\|^2 + \sum_{n=1}^{\infty} b_n^2 \|\sin nx\|^2$$

function

$\|h\|^2 := \langle h, h\rangle.$

$= \int_0^{2\pi} h(x)^2\,dx$

$$= c^2 \underbrace{\int_0^{2\pi} 1^2\,dx}_{2\pi} + \sum_{n=1}^{\infty} a_n^2 \underbrace{\int_0^{2\pi} \cos^2 nx\,dx}_{\pi} + \sum_{n=1}^{\infty} b_n^2 \underbrace{\int_0^{2\pi} \sin^2 nx\,dx}_{\pi}$$

$$= \pi\left(2c^2 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2\right)$$

$\|\vec{v}\|^2$

$|\vec{v}|^2$

$$\boxed{\frac{1}{\pi}\int_0^{2\pi} f(x)^2\,dx = 2c^2 + \sum_{n=1}^{\infty} a_n^2 + \sum_{n=1}^{\infty} b_n^2}$$

$f(x) = x$ on $(0, 2\pi)$, extends periodically.

$$c = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{(2\pi)^2}{2}\cdot\frac{1}{2\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx = \frac{1}{\pi}\left(-\frac{2\pi}{n}\right) = -\frac{2}{n}.$$

Parseval's identity $\Rightarrow$

$$\underbrace{\frac{1}{\pi}\int_0^{2\pi} x^2\,dx}_{\shortparallel} = 2\pi^2 + \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)^2 = 2\pi^2 + 4\zeta(2)$$

$$\frac{1}{\pi}\cdot\frac{(2\pi)^3}{3} = 2\pi^2 + 4\zeta(2) \Rightarrow \zeta(2) = \left(\frac{8}{3}-2\right)\frac{\pi^2}{4}$$

$$= \frac{2}{3}\cdot\frac{\pi^2}{4} = \frac{\pi^2}{6}.$$

Let $f(x) = x^2$, on $(0, 2\pi]$, extends periodically

$$\|f\|^2 = \int_0^{2\pi} f(x)^2\,dx = \int_0^{2\pi} x^4\,dx = \frac{(2\pi)^5}{5}$$

Fourier series: $\displaystyle x^p = \sum_{n=-\infty}^{\infty} c(p,n)\, e^{inx}$

$$c(p,0) = \frac{1}{2\pi}\int_0^{2\pi} x^p\,dx = \frac{(2\pi)^p}{p+1}$$

$$\underset{\substack{\uparrow\\ n\neq 0}}{c(p,n)} = -\left(\frac{(2\pi)^{p-1}}{ni} + \frac{p(2\pi)^{p-2}}{(ni)^2} + \cdots + \frac{p!}{(ni)^p}\right)$$

In particular, $p = 2$:

$$c(2,0) = \frac{(2\pi)^2}{3}$$

$$\underset{\substack{\uparrow\\ n\neq 0}}{c(2,n)} = -\left(\frac{2\pi}{ni} + \frac{2}{(ni)^2}\right) = -\frac{2\pi}{ni} + \frac{2}{n^2}$$

$$\|f\|^2 = \langle f, f\rangle = \int_0^{2\pi} |f(x)|^2\,dx$$

$$= \left\langle \sum_{n=-\infty}^{\infty} c_n e^{inx}, \sum_{m=-\infty}^{\infty} c_m e^{imx}\right\rangle$$

$$= \sum_{n=-\infty}^{\infty} c_n \bar{c}_n \|e^{inx}\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2 \int_0^{2\pi} \cancel{e^{inx} e^{-inx}}\,dx$$

$$= 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2 .$$

$$\langle g, h\rangle = \int_0^{2\pi} g(x)\, \overline{h(x)}\, dx$$

$$\langle ag, bh\rangle = \int_0^{2\pi} a g(x)\, \overline{b\, h(x)}\, dx = a\bar{b} \int_0^{2\pi} g(x)\, \overline{h(x)}\, dx$$

constants $\in \mathbb{C}$

$$= a\bar{b}\, \langle g, h\rangle .$$

$$x^2 = \underbrace{\frac{(2\pi)^2}{3}}_{c_0} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \underbrace{\left(-\frac{2\pi}{ni} + \frac{2}{n^2}\right)}_{c_n \ (n\neq 0)} e^{inx}$$

$$\underbrace{\int_0^{2\pi} (x^2)^2\, dx}_{\|x^2\|^2} = 2\pi\left(c_0^2 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |c_n|^2\right)$$

$$= 2\pi\left(\frac{(2\pi)^4}{3^2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \underbrace{\left|\frac{2\pi i}{n} + \frac{2}{n^2}\right|^2}_{\left(\frac{2\pi}{n}\right)^2 + \left(\frac{2}{n^2}\right)^2}\right)$$

$$= 2\pi\left(\frac{(2\pi)^4}{3^2} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{4\pi^2}{n^2} + \frac{4}{n^4}\right)\right)$$

$$= 2\pi\left(\frac{(2\pi)^4}{3^2} + 2\sum_{n=1}^{\infty} \left(\frac{4\pi^2}{n^2} + \frac{4}{n^4}\right)\right)$$

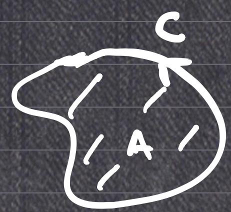
$$\frac{(2\pi)^5}{5} = 2\pi\left(\frac{(2\pi)^4}{9} + 2\cdot 4\pi^2 \cdot \frac{\pi^2}{6} + 8\,\underbrace{\zeta(4)}_{?}\right)$$

$$\zeta(4) = \frac{\pi^4}{90} .$$

$$\langle g, h\rangle := \int_0^{2\pi} g(x)\, \overline{h(x)}\, dx$$

$$\|g\|^2 := \langle g, g\rangle .$$

Isoperimetric inequality.



length = L
Area = A.

$$\frac{L^2}{A} \geq 4\pi$$

equality holds $\Leftrightarrow$ circle.

(by Green's Theorem)

$$A = \frac{1}{2}\int_C x\,dy - y\,dx$$

C: $\vec{r}(t) = (x(t), y(t))$, $a \leq t \leq b$

$dy = y'(t)\,dt$

$dx = x'(t)\,dt$

$$:= \frac{1}{2}\int_{t=a}^{t=b} x(t)y'(t)dt - y(t)x'(t)dt$$

$$= \frac{1}{2}\int_{t=a}^{t=b} \left(x(t)y'(t) - y(t)x'(t)\right)dt$$

Assume : $|\vec{r}'(t)| \equiv 1$, $0 \leq t \leq L$

$x(t)$, $y(t)$ periodic functions with period L.

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \underbrace{e^{i\left(\frac{2\pi}{L}\right)nt}}_{\text{period} = L}$$

$$y(t) = \sum_{n=-\infty}^{\infty} y_n e^{i\left(\frac{2\pi}{L}\right)nt}$$

$$x'(t) = \sum_{n=-\infty}^{\infty} x_n \cdot \frac{2\pi n}{L} i e^{i\left(\frac{2\pi}{L}\right)nt}$$

$$y'(t) = \sum_{n=-\infty}^{\infty} y_n \cdot \frac{2\pi n}{L} i e^{i\left(\frac{2\pi}{L}\right)nt}$$

$$A = \frac{1}{2}\int_0^L x(t)y'(t) - y(t)x'(t)\,dt = \frac{1}{2}\left(\langle x(t), y'(t)\rangle - \langle y(t), x'(t)\rangle\right)$$

$$= \frac{1}{2}\left\langle \sum_{n=-\infty}^{\infty} x_n e^{i\left(\frac{2\pi}{L}\right)nt},\ \sum_{m=-\infty}^{\infty} y_m\left(\frac{2\pi m}{L}\right) i e^{i\left(\frac{2\pi}{L}\right)mt} \right\rangle$$

$$- \frac{1}{2}\left\langle \sum_{n=-\infty}^{\infty} y_n e^{i\left(\frac{2\pi}{L}\right)nt},\ \sum_{n=-\infty}^{\infty} x_n\left(\frac{2\pi n}{L}\right) i e^{i\left(\frac{2\pi}{L}\right)nt} \right\rangle$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} x_n \overline{\left(y_n \cdot \frac{2\pi n}{L} i\right)} \left\| e^{i\left(\frac{2\pi}{L}\right)nt} \right\|^2$$

$$- \frac{1}{2}\sum_{n=-\infty}^{\infty} y_n \overline{\left(x_n \cdot \frac{2\pi n}{L} i\right)} \underbrace{\left\| e^{i\left(\frac{2\pi}{L}\right)nt} \right\|^2}_{= \int_0^L \underbrace{e^{i\left(\frac{2\pi}{L}\right)nt} \cdot e^{-i\left(\frac{2\pi}{L}\right)nt}}_{=1} dt \ = L}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(y_n \bar{x}_n \cdot \frac{2\pi n}{L} i - x_n \bar{y}_n \cdot \frac{2\pi n}{L} i\right) \cdot L$$

$$= \pi i \sum_{n=-\infty}^{\infty} (y_n \bar{x}_n - x_n \bar{y}_n) n$$

$$A = \pi i \sum_{n=-\infty}^{\infty} (y_n \bar{x}_n - x_n \bar{y}_n) n$$

$$= \left| \pi i \sum_{n=-\infty}^{\infty} (y_n \bar{x}_n - x_n \bar{y}_n) n \right|$$

$A \in \mathbb{R}$

$A \geq 0$

$\Rightarrow |A| = A$.

$$\leq \pi \sum_{n=-\infty}^{\infty} |y_n \bar{x}_n - x_n \bar{y}_n| |n|$$

(triangle inequality)

$$\leq \pi \sum_{n=-\infty}^{\infty} |n| \left(|y_n \bar{x}_n| + |x_n \bar{y}_n| \right)$$

$$\leq \pi \sum_{n=-\infty}^{\infty} |n| \cdot \left(\frac{|y_n|^2 + \underbrace{|\bar{x}_n|^2}_{=|x_n|^2}}{2} + \frac{|x_n|^2 + \underbrace{|\bar{y}_n|^2}_{|y_n|^2}}{2} \right)$$

$$= \pi \sum_{n=-\infty}^{\infty} |n| \left(|x_n|^2 + |y_n|^2 \right)$$

$$|\vec{r}\,'(t)|^2 \equiv 1 \rightsquigarrow x'(t)^2 + y'(t)^2 \equiv 1 \Rightarrow \underbrace{\int_0^L (x'(t))^2 + (y'(t))^2 \, dt} = L$$

$$L\left(\underbrace{\sum_{n=-\infty}^{\infty}\left|x_n\left(\frac{2\pi n}{L}\right)i\right|^2}_{\frac{1}{L}\int_0^L (x'(t))^2\,dt}+\underbrace{\sum_{n=-\infty}^{\infty}\left|y_n\left(\frac{2\pi n}{L}\right)i\right|^2}_{\frac{1}{L}\int_0^L (y'(t))^2\,dt}\right)=L$$

$$\Rightarrow \left(\frac{2\pi}{L}\right)^2\sum_{n=-\infty}^{\infty} n^2\left(|x_n|^2+|y_n|^2\right)=1$$

$$A \le \pi\sum_{n=-\infty}^{\infty}|n|\left(|x_n|^2+|y_n|^2\right)$$

$$\le \pi\sum_{n=-\infty}^{\infty}|n|^2\left(|x_n|^2+|y_n|^2\right)=\pi\cdot\frac{L^2}{(2\pi)^2}$$

$$=\frac{L^2}{4\pi}$$

$|n|<n^2$ when $n\neq 0,\pm 1$.

equality holds $\Leftrightarrow |x_n|^2+|y_n|^2=0 \quad \forall n\neq 0,1,-1.$

$\Leftrightarrow x_n=y_n=0 \quad \forall n\neq 0,1,-1.$

$$x(t)=x_{-1}e^{\left(\frac{2\pi}{L}\right)(-1)it}+x_0+x_1e^{\left(\frac{2\pi}{L}\right)it}=\cdots$$

$$y(t)=y_{-1}e^{\left(\frac{2\pi}{L}\right)(-1)it}+y_0+y_1e^{\left(\frac{2\pi}{L}\right)it}=\cdots$$