Solution

Tutorial 12: Fourier Series

Want to solve

$$f(x) = C + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

for periodic function $f: \mathbb{R} \to \mathbb{R}$ ($f(x+2\pi) = f(x)\ \forall x$)

$$\begin{cases} a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\, dx \\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\, dx \\ C = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx \end{cases}$$

See lecture 23

Question: how do we derive these formula?

1 Find the fourier series of the following function

(a) $f(x) = \begin{cases} 2 & x \in [0,\pi) \\ 0 & x \in (-\pi, 0) \end{cases}$

$a_m = \frac{1}{\pi}\int_0^{\pi} 2\cos mx\, dx = \frac{2}{m\pi}\sin mx \Big|_0^{\pi} = 0$

$b_m = \frac{1}{\pi}\int_0^{\pi} 2\sin mx\, dx = -\frac{2}{m\pi}\cos mx \Big|_0^{\pi} = -\frac{2}{m\pi}(\cos m\pi - 1) = \begin{cases} 0 & \text{for even } m \\ \frac{4}{m\pi} & \text{for odd } m \end{cases}$

$C = \frac{1}{2\pi}\int_0^{\pi} 2\, dx = 1$

Fourier series $1 + \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\sin((2k+1)x)$

(b) $f(x) = p + qx$

$C = \frac{1}{2\pi}\int_{-\pi}^{\pi}(p+qx)\, dx = p$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi}(p+qx)\cos mx\, dx$$

$$= \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} p\cos mx\, dx}_{=0 \text{ by calculation}} + \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\underbrace{qx}_{\text{odd}}\ \underbrace{\cos mx}_{\text{even}}}_{\text{odd}}\, dx}_{=0}$$

Hence $a_m = 0$

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi}(p+qx)\sin mx\, dx$$

by calculation & integration by part

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{p\sin mx}_{\text{odd}}\, dx + \frac{1}{\pi}\int_{-\pi}^{\pi} qx\sin mx\, dx = q\frac{2(-1)^m}{m}$$

Hence fourier Series $= p + \sum_{m=1}^{\infty} \frac{2(-1)^m}{m} \sin(mx)$

(c) $f(x) = \sin(ax)$ on $(-\pi, \pi)$

$c = 0$ (as f is odd)

$a_n = 0$ (as f is odd)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin ax \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos((a-n)x) - \cos((a+n)x)\, dx$$

$$= \left[\frac{-\sin((a-n)x)}{2\pi(a-n)} + \frac{\sin((a+n)x)}{2\pi(a+n)} \right]_{-\pi}^{\pi}$$

$$= -\frac{\sin((a-n)x)}{\pi(a-n)} \quad \frac{\sin((a+n)\pi)}{\pi(a+n)}$$

2 let $f: \mathbb{R} \to \mathbb{R}$ be s.t. $f(x+\pi) = -f(x)$

if $f(x) = b_0 + \sum_{k=1}^{\infty}(a_n \cos(nx) + b_n \sin(nx))$, then $\forall$ even n,

$a_n = b_n = 0$.

$$a_{2n} = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(2mx)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} -f(x+\pi)\cos(2n(x+\pi))\, dx$$

let $y = x+\pi$

$$= -\frac{1}{\pi}\int_{\pi}^{3\pi} f(y)\cos(2ny)\, dy = -a_{2n}$$

similarly for $b_{2n} = -b_{2n}$

remark: trigonometry equality; odd-even function