$f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}.$

then

$$f(x) = c + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where $c = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} (\cos nx) f(x)\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} (\sin nx) f(x)\,dx.$$

$\vec{v}, \vec{w} \in \mathbb{R}^n$, $\quad \vec{v} = \sum_{i=1}^{n} v_i \hat{e}_i \quad , \quad \vec{w} = \sum_{i=1}^{n} w_i \hat{e}_i.$

$$\hat{e}_i = (0, \ldots, \underset{i}{1}, \ldots\ldots 0)$$

$$\vec{v}\cdot\hat{e}_j = \left(\sum_{i=1}^{n} v_i \hat{e}_i\right)\cdot\hat{e}_j = \sum_{i=1}^{n} v_i \delta_{ij}$$

$$= v_j$$

$$\hat{e}_i \cdot \hat{e}_j = (0, \ldots, \underset{i}{1}, \ldots, 0) \cdot (0, \ldots, \underset{j}{1}, \ldots 0) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases} = \delta_{ij}$$

$$\vec{v} = \sum_{i=1}^{n} (\vec{v}\cdot\hat{e}_i)\hat{e}_i.$$

$$\vec{v}\cdot\vec{w} = \sum_{i=1}^{n} v_i w_i$$

$\langle \vec{v}, \vec{w} \rangle$

$f, g : [0, 2\pi] \to \mathbb{R}$

$$\langle f, g \rangle := \int_0^{2\pi} f(x) g(x)\,dx$$

$\langle f, g+h \rangle = \langle f, g \rangle + \langle f, h \rangle \quad$ (c.f. $\vec{v}\cdot(\vec{u}+\vec{w}) = \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{w}$)

$$\langle \cos mx, \cos nx \rangle = \begin{cases} \pi & \text{if } m=n \in \mathbb{N} \\ 0 & \text{if } m\neq n \end{cases}$$

$$\left\langle \frac{1}{\sqrt{\pi}}\cos mx, \frac{1}{\sqrt{\pi}}\cos nx \right\rangle = \begin{cases} 1 & \text{if } m=n \in \mathbb{N} \\ 0 & \text{if } m\neq n. \end{cases} = \delta_{mn}.$$

$$f(x) = c + \sum_{n=1}^{\infty} \tilde{a}_n \frac{1}{\sqrt{\pi}} \cos nx + \sum_{n=1}^{\infty} \tilde{b}_n \cdot \frac{1}{\sqrt{\pi}} \sin nx$$

$$\tilde{a}_n = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \cos nx \, dx = \left\langle f, \frac{1}{\sqrt{\pi}} \cos nx \right\rangle .$$

$$\tilde{b}_n = \frac{1}{\sqrt{\pi}} \int_0^{2\pi} f(x) \sin nx \, dx = \left\langle f, \frac{1}{\sqrt{\pi}} \sin nx \right\rangle$$

$$c = \frac{1}{2\pi} \int_0^{2\pi} f(x) \, dx = \left\langle f(x), \frac{1}{2\pi} \right\rangle ..$$

$\{e_1, e_2, \ldots\}$ functions on $[0, 2\pi]$

if $\langle e_i, e_j \rangle = \delta_{ij}$

$$\int_0^{2\pi} e_i(x) e_j(x) = \delta_{ij}$$

$$\vec{v} \cdot \vec{v} = |\vec{v}|^2 .$$

$f, g : [0, 2\pi] \to \mathbb{C}$

$\overline{a+bi} = a - bi .$

$$\langle f, g \rangle = \int_0^{2\pi} f(x) \overline{g(x)} \, dx$$

$$\langle f, f \rangle = \int_0^{2\pi} f(x) \overline{f(x)} \, dx = \int_0^{2\pi} |f(x)|^2 dx \geq 0 .$$

$$z \bar{z} = |z|^2 .$$

$\{e^{inx}\}_{n \in \mathbb{Z}} .$

$e^{ix} := \cos x + i \sin x .$

$$\langle e^{inx}, e^{imx} \rangle = \int_0^{2\pi} e^{inx} \overline{e^{imx}} \, dx$$

$$\overline{e^{ix}} = \cos x - i \sin x$$

$$= \cos(-x) + i \sin(-x)$$

$$= e^{-ix} .$$

$$= \int_0^{2\pi} e^{inx} \cdot e^{-imx} \, dx$$

$$= \int_0^{2\pi} e^{i(n-m)x} \, dx$$

$$= \begin{cases} \int_0^{2\pi} 1\,dx = 2\pi & \text{if } m = n \in \mathbb{Z}. \\ \underbrace{\left[\frac{1}{(n-m)i} e^{i(n-m)x}\right]_0^{2\pi}} = 0 & \text{if } m \neq n \end{cases}$$

$$= \frac{1}{(n-m)i}\left(\underbrace{e^{i(n-m)2\pi}}_{\cos 2(n-m)\pi + i \sin 2(n-m)\pi} - 1\right)$$

$$= 1 + 0i = 1$$

$$\left\langle \frac{1}{\sqrt{2\pi}} e^{inx}, \frac{1}{\sqrt{2\pi}} e^{imx} \right\rangle = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases} = \delta_{mn}$$

$$f(x) = c + \sum_{n=1}^{\infty} a_n \underbrace{\cos nx}_{\frac{e^{inx}+e^{-inx}}{2}} + \sum_{n=1}^{\infty} b_n \underbrace{\sin(nx)}$$

$$\sin(nx) = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$$

$$\left(= \frac{\cos nx + \cancel{i \sin nx} + \cos nx - \cancel{i \sin nx}}{2} \right)$$

$$f(x) = \sum_{n=-\infty}^{\infty} \underbrace{c_n}_{?} e^{inx} = \sum_{n=-\infty}^{\infty} \tilde{c}_n \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{inx}}$$

$$\left\langle \frac{1}{\sqrt{2\pi}} e^{inx}, \frac{1}{\sqrt{2\pi}} e^{imx} \right\rangle = \delta_{mn}.$$

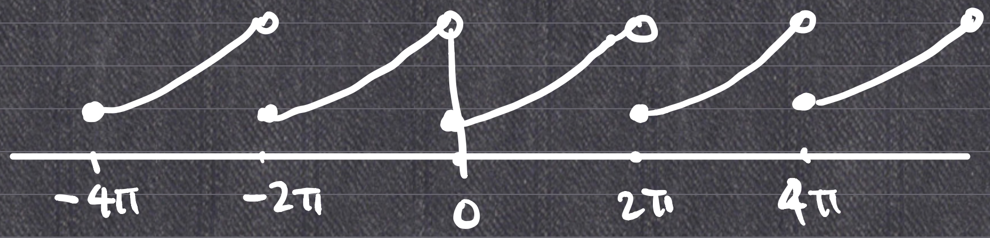
$$\tilde{c}_n = \left\langle f, \frac{1}{\sqrt{2\pi}} e^{inx} \right\rangle$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(x) e^{-inx}\,dx.$$

$$c_n = \frac{\tilde{c}_n}{\sqrt{2\pi}} = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\,dx.$$

e.g. $f(x) = e^x$ on $[0, 2\pi)$

and $f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}.$



$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\overline{e^{inx}}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^x e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{x-inx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{(1-in)x}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-in}e^{(1-in)x}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{1-in}\Big(\underbrace{e^{(1-in)2\pi}}_{\shortparallel} - 1\Big) = \frac{e^{2\pi}-1}{2\pi(1-in)}$$

$$e^{2\pi - i2\pi n} = e^{2\pi}\underbrace{e^{-2\pi n i}}$$

$$\therefore\ f(x) \text{ "="} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi}-1}{2\pi(1-in)} e^{inx}.$$

$$= e^{2\pi}(\cos 2\pi n - i\sin 2\pi n)$$
$$= e^{2\pi}\cdot 1.$$

e.g. $f(x) = \dfrac{a\sin x}{1-2a\cos x + a^2}$ $\qquad |a| < 1.$

$$= \frac{a\left(\dfrac{e^{ix}-e^{-ix}}{2i}\right)}{1 - \cancel{2}a\cdot\dfrac{e^{ix}+e^{-ix}}{\cancel{2}} + a^2} = \frac{a}{2i}\cdot\frac{e^{ix}-e^{-ix}}{1-ae^{ix}-ae^{-ix}+a^2}$$

$$= \frac{a}{2i} \cdot \frac{e^{ix} - e^{-ix}}{(1-ae^{ix})(1-ae^{-ix})}$$

$$\frac{1}{1-z} \overset{|z|<1}{=} \sum_{n=0}^{\infty} z^n .$$

$$= \frac{1}{2i}\left(\frac{1}{1-ae^{ix}} - \frac{1}{1-ae^{-ix}} \right)$$

$$|ae^{ix}| = \underbrace{|a|}_{<1}\,\underbrace{|e^{ix}|}_{1}$$

$$= \frac{1}{2i}\left(\sum_{n=0}^{\infty} (ae^{ix})^n - \sum_{n=0}^{\infty} (ae^{-ix})^n \right)$$

$$= \frac{1}{2i}\left(\sum_{n=0}^{\infty} a^n e^{inx} - \sum_{n=0}^{\infty} a^n e^{-inx} \right)$$

$$\int_0^{2\pi} \frac{a \sin x}{1-2a\cos x + a^2}\,dx = \int_0^{2\pi} \frac{1}{2i}\left(\sum_{n=0}^{\infty} a^n e^{inx} - \sum_{n=0}^{\infty} a^n e^{-inx} \right) dx$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} a^n \underbrace{\int_0^{2\pi} e^{inx}\,dx} - \frac{1}{2i} \sum_{n=0}^{\infty} a^n \int_0^{2\pi} e^{-inx}\,dx$$

$$= \begin{cases} \cancel{\left[\frac{1}{in} e^{inx}\right]_0^{2\pi}} \ 0 & \text{if } n \neq 0 \\ 2\pi & \text{if } n = 0 \end{cases}$$

$$= \frac{1}{2i}\left(a^0 \cdot 2\pi - a^0 \cdot 2\pi \right) = 0 .$$

$$e^{ix} = \cos x + i \sin x .$$

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\cdot\frac{x^3}{3!} + \frac{x^4}{4!} + i\,\frac{x^5}{5!} + \cdots$$

$$= \left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)$$

$$= \cos x + i\sin x.$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$

$$e^{i\pi}+1=0$$

$$\left(e^{\frac{2\pi i}{3}}\right)^3 = e^{\frac{2\pi i}{3}\cdot 3} = e^{2\pi i} = 1.$$

$$\left(e^{\frac{4\pi i}{3}}\right)^3 = e^{4\pi i} = 1$$