Tutorial 11

1 radius of converge (of $\sum_{n=1}^{\infty} a_n z^n$) $= \dfrac{1}{\lim_n \sqrt[n]{|a_n|}} = \lim_{n\to\infty} \left|\dfrac{a_n}{a_{n+1}}\right|$

↑ def in lecture ↑ ratio test

(a) $\sum_{n=1}^{\infty} n^n z^n$

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n^n}} = \frac{1}{\lim_{n\to\infty} n} = 0$$

(b) $\sum_{n=1}^{\infty} n^2 z^n$

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n^2}} = \frac{1}{(\lim_{n\to\infty} \sqrt[n]{n})^2} = \frac{1}{1^2} = 1$$

(c) $\sum_{n=1}^{\infty} \frac{3^n}{n!} z^n$

$$R = \lim_{n\to\infty} \left(\frac{3^n}{n!} \frac{(n+1)!}{3^{n+1}}\right) = \lim_{n\to\infty} \frac{n}{3} = \infty$$

(d) $\sum_{n=1}^{\infty} \frac{2^n}{n^2} z^n$

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}}} = \frac{1}{2}$$

(e) $\sum_{n=1}^{\infty} \frac{n^3}{3^n} z^n$

$$R = \lim_{n\to\infty} \left(\frac{n^3}{3^n} \frac{3^{n+1}}{(n+1)^3}\right) = 3.$$

(f) $\sum_{n=1}^{\infty} \frac{n^n}{n!} z^n$

$$R = \lim_{n\to\infty} \frac{n^n}{n!} \frac{(n+1)!}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{(\frac{1}{n}+1)^n} = \frac{1}{e}$$

2 Suppose that $\sum_{n=0}^{\infty} C_n (x-a)^n$ converges at $x=b$ where $a, b \in \mathbb{R}$, $b<a$. for what value of x the series must converges or converges absolutely?

Ans: Converge absolutely on $(b, 2a-b)$ and converges on $[b, 2a-b)$. No information on $\mathbb{R}\setminus[b, 2a-b)$

Sol: $\sum_{n=0}^{\infty} C_n (b-a)^n$ converges

$$\lim_{n\to\infty} C_n (b-a)^n = 0 \Rightarrow |C_n (b-a)^n| \leq M$$

$$\forall x \in (b, 2a-b),\ |x-a| < |b-a|$$

$$|(C_n (x-a)^n)| = |C_n (b-a)^n| \left|\frac{x-a}{b-a}\right|^n \leq M r^n \text{ where } 0 \leq \left|\frac{x-a}{b-a}\right| =: r < 1$$

Hence $\sum C_n (x-a)^n$ converge absolutely on $(b, 2a-b)$

3 Let $\{C_n\}$ be a infinite sequence of integers s.t. there are infinitely many $n \in \mathbb{N}$ where $C_n \neq 0$

(a) Show the radius of convergence of $\sum_{n=0}^{\infty} C_n z^n$ is at most 1

pf: Assume $R > 1$, $\sum |C_n 1^n|$ converge, $\lim_{n\to\infty} C_n = 0$

contradiction to there are infinitely many non-zero integer C_n

(b) Tayler series of a smooth function with radius of convergence 1

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \quad \text{with } R = 1$$

(c)
$$\frac{1}{1-2z} = \sum_{n=0}^{\infty} (2z)^n \Rightarrow \text{Take } C_n = 2^n$$

(d) Take $C_n = \pi^n$

$$\lim_{n\to\infty} \frac{C_n}{C_{n+1}} = \lim_{n\to\infty} \frac{\pi^n}{\pi^{n+1}} = \lim_{n\to\infty} \frac{1}{4}$$

Appendix let $R = \lim_{n\to\infty} \left| \frac{C_n}{C_{n+1}} \right|$

By ratio test, series converge if

$$\lim_{n\to\infty} \frac{|C_{n+1}(z-a)^{n+1}|}{|C_n(z-a)^n|} < 1$$

$$\Leftrightarrow |z-a| < \frac{1}{\lim_{n\to\infty} \frac{|C_{n+1}|}{|C_n|}} = \lim_{n\to\infty} \left| \frac{C_n}{C_{n+1}} \right|$$