Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Fourier series

$f : \mathbb{R} \to \mathbb{R}$, $f(x+2\pi) = f(x) \ \forall x \in \mathbb{R}$

periodic function.

Want: $f(x) = c + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$ — $(*)$

- Suppose $(*)$ holds, then $c = ?$, $a_n = ?$, $b_n = ?$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \int_0^{2\pi} \sin mx \sin nx \, dx$$

$$= \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \qquad (m, n \in \mathbb{N})$$

$$\int_0^{2\pi} \cos mx \sin nx \, dx = 0 \quad \forall m, n \in \mathbb{N}.$$

Proof: $\int_0^{2\pi} \cos mx \cos nx \, dx = \int_0^{2\pi} \frac{1}{2}\left(\cos(m+n)x + \cos(m-n)x\right) dx$

$$= \begin{cases} \frac{1}{2}\int_0^{2\pi} (\cos 2mx + 1) \, dx & \text{if } m = n \\ \frac{1}{2}\left[\frac{1}{m+n}\sin(m+n)x + \frac{1}{\underbrace{m-n}_{\neq 0}}\sin(m-n)x\right]_0^{2\pi} & \text{if } m \neq n \end{cases}$$

$$= \begin{cases} \frac{1}{2}\left[\frac{1}{2m}\sin 2mx + x\right]_0^{2\pi} = \pi & \text{if } m = n \\ 0. & \text{if } m \neq n \end{cases}$$

Similar for others.

Let $f(x) = c + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$f(x)\cos mx = c\cdot\cos mx + \sum_{n=1}^{\infty} a_n \cos nx \cos mx + \sum_{n=1}^{\infty} b_n \sin nx \cos mx$$

$$\int_0^{2\pi} f(x)\cos mx\,dx = \int_0^{2\pi} c\cdot\cos mx\,dx + \sum_{n=1}^{\infty}\int_0^{2\pi} a_n \underline{\cos nx\cos mx\,dx}$$

$(m \in \mathbb{N})$

$$+ \sum_{n=1}^{\infty}\int_0^{2\pi} b_n \sin nx \cos mx\,dx$$

$$= 0 + \sum_{n=1}^{\infty} a_n \underline{\pi\delta_{mn}} + \sum_{n=1}^{\infty} 0$$

$\delta_{mn} = \begin{cases} 1 & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases}$ $\qquad = a_m \pi$

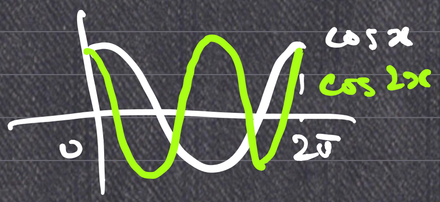
$$\Rightarrow \quad a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\,dx \qquad \forall m \in \mathbb{N}.$$

Similarly, $b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx$.

$c = ?$

$$\int_0^{2\pi} f(x)\,dx = \int_0^{2\pi} c\,dx + \sum_{n=1}^{\infty}\underbrace{\int_0^{2\pi} a_n \cos nx\,dx}_{=0} + \sum_{n=1}^{\infty}\underbrace{\int_0^{2\pi} b_n \sin nx\,dx}_{=0}$$

$$= 2\pi c$$



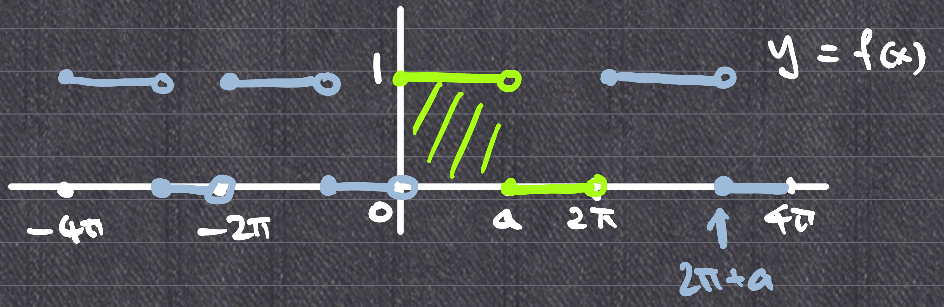
$$\Rightarrow \quad c = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx.$$

To conclude:

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx\right)\cos nx + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx\right)\sin nx$$

e.g. $f(x) = \begin{cases} 1 & \text{if } x \in [0,a) \\ 0 & \text{if } x \in [a, 2\pi) \end{cases}$ $\quad a \in (0, 2\pi)$

and extends periodically with period 2π.



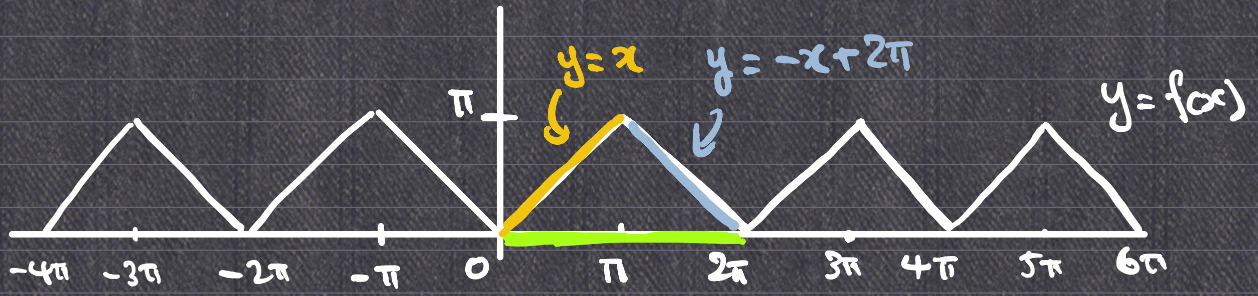
$$C = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi} a.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{a} 1\cdot\cos nx\,dx = \frac{\sin(na)}{n\pi}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{a} 1\cdot\sin nx\,dx = \frac{1-\cos na}{n\pi}$$

$$f(x) \text{ "="} \frac{a}{2\pi} + \sum_{n=1}^{\infty} \underbrace{\frac{\sin(na)}{n\pi}}_{a_n} \cos nx + \sum_{n=1}^{\infty} \underbrace{\left(\frac{1-\cos na}{n\pi}\right)}_{b_n} \sin nx.$$

e.g.



$$C = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\left(\frac{1}{2}\cdot 2\pi\cdot\pi\right) = \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left(\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{n\pi \cancel{\sin(n\pi)} + \cos(n\pi) - 1}{n^2} + \frac{\cos(n\pi) - 1}{n^2}\right)$$

$$= \frac{1}{\pi}\left(\frac{2\cos n\pi}{n^2} - \frac{2}{n^2}\right) = \begin{cases} 0 & \text{if } n \text{ even} \\ -\dfrac{4}{\pi n^2} & \text{if } n \text{ odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \underbrace{f(x)}_{\text{even}} \sin nx \, dx \underset{\text{let } y = x-\pi}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{f(y+\pi)\sin(ny+n\pi)}_{F(y)} \, dy$$

odd.

$$\begin{aligned} F(-y) &= \underbrace{f(-y+\pi)}\sin(-ny+n\pi) \\ &= f(y-\pi)\left(-\sin(ny-n\pi)\right) \\ &= f(y-\pi+2\pi)\left(-\sin(ny-n\pi+2n\pi)\right) \\ &= f(y+\pi)\left(-\sin(ny+n\pi)\right) \\ &= -F(y). \end{aligned}$$

$$\Rightarrow \quad b_n = 0 \quad \forall n \in \mathbb{N}.$$

$$\therefore \; f(x) = \frac{\pi}{2} - \sum_{n \text{ odd}} \frac{4}{n^2\pi}\cos nx$$

$$= \frac{\pi}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi}\cos(2k-1)x.$$

Cor: $\zeta(2) = \dfrac{\pi^2}{6}$.

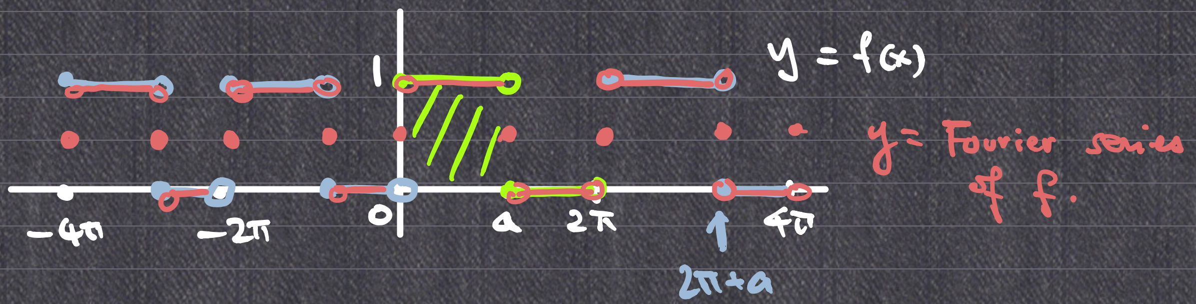
Proof: $\underbrace{f(0)}_{=0} = \dfrac{\pi}{2} - \displaystyle\sum_{k=1}^{\infty}\frac{4}{(2k-1)^2\pi}$

$$\Rightarrow \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi}{2} \Rightarrow \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n \text{ odd}} \frac{1}{n^2} + \sum_{n \text{ even}} \frac{1}{n^2} = \frac{\pi^2}{8} + \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots\right)$$

$$\zeta(2) = \frac{\pi^2}{8} + \frac{1}{2^2}\underbrace{\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)}_{\zeta(2)}$$

$$\frac{3}{4}\zeta(2) = \frac{\pi^2}{8} \Rightarrow \zeta(2) = \frac{\pi^2}{8} \cdot \frac{4}{3} = \frac{\pi^2}{6} \quad \blacksquare.$$



Stein's Fourier Analysis.

Terence Tao — Charles Fefferman