Tutorial 10

Q1: Let $\{a_n\}$ be the Fibonacci Series

i.e. $a_0 = 1,\ a_1 = 1,\ a_2 = 2, \ldots, a_{n+2} = a_{n+1} + a_n$

let $\{b_n\}$ be a sequence s.t. $b_n = \frac{1}{a_n}$

Given $a_k = \frac{1}{\sqrt{5}}(\beta^n - \alpha^n)$ where $\alpha = \frac{1-\sqrt{5}}{2}$ and $\beta = \frac{1+\sqrt{5}}{2}$

Show $\sum_0^\infty b_n$ converges

pf: $$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{\beta^n - \alpha^n}{\beta^{n+1} - \alpha^{n+1}} = \lim_{n\to\infty} \frac{1 - (\frac{\alpha}{\beta})^n}{\beta - \alpha(\frac{\alpha}{\beta})^n} \underset{|\frac{\alpha}{\beta}|<1}{=} \frac{1}{\beta} < 1$$

By ratio test //

Q2 (a), (b), (c): we can rearrange as the series absolute converge

(d) i.e. odd term exchange with even term

Consider $B_N := \sum_{n=1}^N b_{\alpha(n)}$

$$\lim_N B_N = \lim_N A_N + \lim_N (B_N - A_N) \text{ if the limit at rhs exist}$$

$$B_N - A_N = \begin{cases} 0 & \text{if } N \text{ is even} \\ \frac{1}{N+1} - \frac{1}{N} & \text{if } N \text{ is odd} \end{cases}$$

Hence $\lim_N (B_N - A_N) = 0$

$$\lim_N B_N = \lim_N A_N \quad //$$

Key observation

(e)
n	1	2	3	4	5	6	7	$\cdots$
$\beta(n)$	1	2	4	3	6	7	5	

negative

Observation: $-1 + \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \frac{1}{6} + \frac{1}{8} - \frac{1}{5} + \frac{1}{10} + \frac{1}{12} \cdots$

let $S_N = \sum_{n=1}^N b_{\beta(n)}$

Consider $S_{3N} = \sum_{n=1}^{3N} b_{\beta(n)}$

$$= \sum_{k=0}^{N-1} b_{\beta(3k+1)} + b_{\beta(3k+2)} + b_{\beta(3k+3)}$$

$$= \sum_{k=0}^{N-1} b_{2k+1} + b_{4k+2} + b_{4k+4}$$

$$= \sum_{k=0}^{N-1} -\frac{1}{8k^2+12k+4}$$ (negative i.e. S_N decreasing)

$$\geq \sum_{k=0}^{N-1} -\frac{1}{k^2+2k+1}$$ (converge)

By monotone convergence thm, $\lim_{N\to\infty} S_{3N}$ exist, say, its limit is L

$$S_{3N+1} = S_{3N} - \frac{1}{2N+1}$$

$$S_{3N+2} = S_{3N} + \frac{1}{4N+2}$$

$$\lim_{N\to\infty} S_{3N+1} = \lim_{N\to\infty} S_{3N+2} = \lim_{N\to\infty} S_{3N} = L$$

Hence $\lim_{N\to\infty} S_N = L$

(using ε language, u can see so)

$\forall \varepsilon > 0, \exists N_1, N_2, N_3$ s.t. $\forall n > 3\max(N_1, N_2, N_3)$

$|S_n - L| < \varepsilon$ as $n = 3N+1$ or $n = 3N+2$ or $n = 3N+3$ for some N

Q3 Consider $C_k = \frac{f^{k}(0)}{k!}$

want to show $\sum_{k=0}^{\infty} C_k x^k$ converges $\forall x \in (-c, c)$ implies $\sum_{k=0}^{\infty} C_k x^k$ converges absolutely $\forall x \in (-c, c)$

$\forall x_0 \in (-c, c)$, Pick $x_1 \in (-c, c)$ s.t. $|x_1| > |x_0|$

$$\sum_{k=0}^{\infty} |C_k x_0^k| = \sum_{k=0}^{\infty} |C_k x_1^k| \left|\frac{x_0}{x_1}\right|^k \leq M \underbrace{\sum_{k=0}^{\infty} \left|\frac{x_0}{x_1}\right|^k}_{\text{converge as } \left|\frac{x_0}{x_1}\right| < 1} < \infty$$

where $M := \sup_k |C_k x_1|^k < \infty$

as $\sum_{k=0}^{\infty} C_k x_1^k$ converges

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