- Alternating Series Test:

If $a_n \searrow$ and $\lim_{n\to\infty} a_n = 0$,
then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges

$= a_1 - a_2 + a_3 - a_4 + \dots$

Cor: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges when $p > 0$.

$\left(a_n = \frac{1}{n^p}\right)$

- Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(caution: converse is not true)

Proof: $0 \le a_n + |a_n| \le 2|a_n|$

$\uparrow$

$-a_n \le |a_n|$

$\sum_{n=1}^{\infty} 2|a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} (a_n + |a_n|)$ converges

(comparison test)

$$\lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} (a_n + |a_n| - |a_n|)$$

$$= \lim_{N\to\infty} \sum_{n=1}^{N} (a_n + |a_n|) - \lim_{N\to\infty} \sum_{n=1}^{N} |a_n|$$

$$= \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$$

$\therefore \sum_{n=1}^{\infty} a_n$ converges.

Ratio test;

$$\lim_{n\to\infty}\underbrace{\left|\frac{a_{n+1}}{a_n}\right|}_{\frac{|a_{n+1}|}{|a_n|}} = L<1 \Rightarrow \sum_{n=1}^{\infty}|a_n| \text{ converges.}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Root test

$$\lim_{n\to\infty}\sqrt[n]{|a_n|} = r<1 \Rightarrow \sum_{n=1}^{\infty}|a_n| \text{ converges}$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges.}$$

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges (alternating series test)

but $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

e.g. $\sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^n$: Determine the range of x s.t. the series converges.

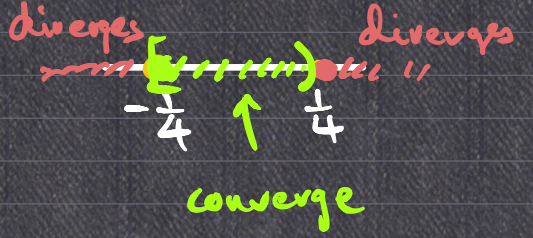
$$\lim_{n\to\infty}\left|\frac{\frac{(2n+2)!}{[(n+1)!]^2}x^{n+1}}{\frac{(2n)!}{(n!)^2}x^n}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)^2}|x| = 4|x|$$

When $4|x|<1$, ratio test & abs convergent test

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^n \text{ converges.}$$

When $4|x|>1$, ratio test $\Rightarrow \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^n$ diverges.

$$\left|\frac{a_{n+1}}{a_n}\right| > 1+\varepsilon \Rightarrow \lim_{n\to\infty}|a_n| = +\infty.$$



When $x = \frac{1}{4}$,

$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \frac{1}{4^n}$ diverges. (from previous lecture)

When $x = -\frac{1}{4}$

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} \left(-\frac{1}{4}\right)^n = \sum_{n=1}^{\infty} (-1)^n \cdot \underbrace{\frac{(2n)!}{(n!)^2} \cdot \frac{1}{4^n}}_{b_n}$$

Stupid to show $b_{n+1} - b_n \leq 0$.

Smarter to show $\frac{b_{n+1}}{b_n} \leq 1$.

$$\frac{\dfrac{(2n+2)!}{[(n+1)!]^2 \cdot 4^{n+1}}}{\dfrac{(2n)!}{(n!)^2 4^n}} = \frac{(2n+2)(2n+1)}{(n+1)^2} \cdot \frac{1}{4}$$

$$= \frac{2n+1}{2n+2} < 1 \Rightarrow b_{n+1} \leq b_n \quad \forall n.$$

$$\frac{b_{n+1}}{b_n} = \frac{2n+1}{2n+2} = 1 - \frac{1}{2n+2} = 1 - \underbrace{\frac{\frac{1}{2}}{n+1}}$$

$$\frac{\dfrac{1}{(n+1)^p}}{\dfrac{1}{n^p}} = 1 - \frac{p}{n+1} + o\left(\frac{1}{n}\right) \text{ as } n \to \infty.$$

Choose $p = 0.4 > \frac{1}{2} \Rightarrow \frac{b_{n+1}}{b_n} = 1 - \frac{\frac{1}{2}}{n+1} < 1 - \frac{0.4}{n+1} + o\left(\frac{1}{n}\right)$ for n large.

$\frac{b_{n+1}}{b_n} < \frac{1}{(n+1)^{0.4}} \Big/ \frac{1}{n^{0.4}}$ $\forall n >> 1$. $(n \geq N)$

$$\Rightarrow \underset{\substack{\downarrow\\ 0}}{0} \le \underset{\substack{\Downarrow\\ 0}}{b_n} < \underset{\substack{\downarrow\\ 0}}{C\cdot\frac{1}{n^{0.4}}}$$

$\therefore \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^4$ converges when $x = -\frac{1}{4}$ (alternating series test) $\square$

e.g. $$\sum_{n=1}^{\infty} \frac{\sin(n)}{n} = \sum_{n=1}^{\infty} \underbrace{a_n}_{\sin(n)} \underbrace{b_n}_{\frac{1}{n}} = \lim_{N\to\infty} \sum_{n=1}^{N} a_n b_n \text{ converges.}$$

Summation by parts:

$$\sum_{n=1}^{N+1} a_n b_n = b_{N+1} \sum_{n=1}^{N+1} a_n - \sum_{n=1}^{N} \underbrace{\left(\sum_{k=1}^{n} a_k\right)} \underbrace{(b_{n+1} - b_n)}$$

$$\int f g \, dx = F g - \int F g' \, dx$$

let $F' = f$

- Dirichlet test:
  - $\left|\sum_{n=1}^{N} a_n\right| \le C$,
  - $b_n \searrow$, $\lim_{n\to\infty} b_n = 0$

  $\Rightarrow \sum_{n=1}^{\infty} a_n b_n$ converges.

$$\sum_{n=1}^{N} \left| \underbrace{\left(\sum_{k=1}^{n} a_k\right)} (b_{n+1} - b_n) \right| \le C \sum_{n=1}^{N} |b_{n+1} - b_n|$$

$$= C \sum_{n=1}^{N} (b_n - b_{n+1})$$

$$= C(b_1 - b_{N+1}) \le C b_1$$

$$\sum_{n=1}^{N} \left| \left(\sum_{k=1}^{n} a_k\right)(b_{n+1}-b_n) \right| \quad \text{bounded}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left| \left(\sum_{k=1}^{n} a_k\right)(b_{n+1}-b_n) \right| \quad \text{converges.}$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_k\right)(b_{n+1}-b_n) \quad \text{converges.}$$

Sum-by-parts $\Longrightarrow$ $\sum_{n=1}^{\infty} a_n b_n$ converges.

Abel's test:

$$\left\{\begin{array}{l} \sum_{n=1}^{\infty} a_n \text{ converges} \\ b_n \text{ is monotone and} \\ \quad \text{bounded} \end{array}\right. \Longrightarrow \sum_{n=1}^{\infty} a_n b_n \text{ converges.}$$

e.g. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n}$ $\qquad a_n = \sin(n)$, $b_n = \frac{1}{n} \searrow 0.$

Try Dirichlet test:

<u>Need</u>: $\left| \sum_{n=1}^{N} \sin(n) \right| \le C.$

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i} \qquad \left\{\begin{array}{l} e^{xi} = \cos x + i\sin x \\ e^{-xi} = \cos x - i \sin x \end{array}\right.$$

$$\sum_{n=1}^{N} \sin(n) = \sum_{n=1}^{N} \frac{e^{in}-e^{-in}}{2i}$$

$$= \frac{1}{2i}\left(\sum_{n=1}^{N} (e^{i})^n - \sum_{n=1}^{N} (e^{-i})^n\right)$$

$$= \frac{1}{2i}\left(\frac{e^i(1-(e^i)^N)}{1-e^i} - \frac{e^{-i}(1-(e^{-i})^N)}{1-e^{-i}}\right)$$

$$= \frac{1}{2}\left(\frac{e^i - e^{i(N+1)}}{1-e^i} - \frac{1-e^{-iN}}{e^i-1}\right)$$

$$= \frac{1}{2i}\cdot\frac{e^i - e^{i(N+1)} + 1 - e^{-iN}}{1-e^i}$$

$$\left|\sum_{n=1}^{N}\sin(n)\right| = \frac{1}{2}\,\frac{|e^i - e^{i(N+1)} + 1 - e^{-iN}|}{|1-e^i|}$$

$$\le \frac{1}{2}\cdot\frac{1}{|1-e^i|}\left(|e^i| + |e^{-i(N+1)}| + 1 + |e^{-iN}|\right)$$

$$\le \frac{1}{2}\cdot\frac{4}{|1-e^i|}$$

Dirichlet test $\Rightarrow$ $\sum_{n=1}^{\infty}\frac{\sin(n)}{n}$ converges.

But $\sum_{n=1}^{\infty}\left|\frac{\sin(n)}{n}\right|$ diverges.

because.

$|\sin(n_k)| \ge \cos\frac{1}{2}$

$$\sum_{n=1}^{\infty}\left|\frac{\sin(n)}{n}\right| \ge \sum_{k=1}^{\infty}\left|\frac{\sin(n_k)}{n_k}\right| \ge \sum_{k=1}^{\infty}\frac{\cos\frac{1}{2}}{(k-1)\pi+\frac{\pi}{2}+\frac{1}{2}} \sim \sum_{k=1}^{\infty}\frac{1}{k\pi}$$

diverges.

$$\therefore \sum_{n\geq 1}^{\infty} \left| \frac{\sin(n)}{n} \right| \quad \text{diverges.}$$

Definitions:

$\sum a_n$ converges absolutely

$\overset{\text{def}}{\iff} \sum |a_n|$ converges (hence $\sum a_n$ converges)

$\sum a_n$ converges conditionally

$\overset{\text{def}}{\iff} \sum a_n$ converges but $\sum |a_n|$ diverges