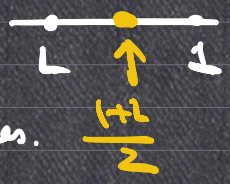
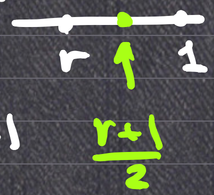
$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1 \rightsquigarrow \frac{a_{n+1}}{a_n} < \underbrace{\frac{1+L}{2}} < 1$. for large n.

$\Rightarrow a_n \le \underbrace{C\left(\frac{1+L}{2}\right)^n}$



$\underbrace{\sum\left(\frac{1+L}{2}\right)^n}_{\text{converges}} \Rightarrow \sum a_n$ converges.

$\lim_{n\to\infty} \sqrt[n]{a_n} = r < 1 \longrightarrow \sqrt[n]{a_n} < \underbrace{\frac{r+1}{2}}$ for large n.



$\Rightarrow a_n < \underbrace{\left(\frac{r+1}{2}\right)^n}_{b_n} \quad \forall n \gg 1$

$\because \sum b_n$ converges $\left(\frac{r+1}{2} < 1\right)$

$\Rightarrow \sum a_n$ converges.

e.g. $\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n}\right)^{-n^2}}_{a_n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \left\{\left(1+\frac{1}{n}\right)^{-n^2}\right\}^{\frac{1}{n}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

$\therefore \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$.

Ratio Test (1024 version):

Given $a_n, b_n > 0 \ \forall n \in \mathbb{N}$, and $\exists N > 0$ s.t.

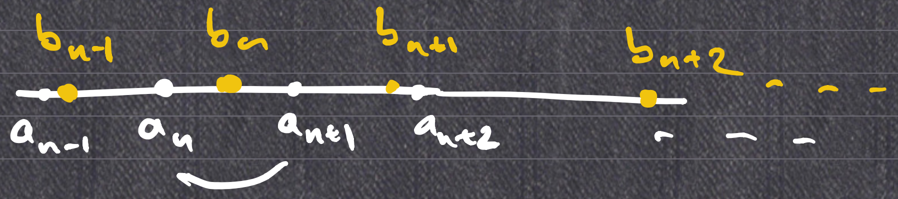
$$\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n} \quad \forall n \ge N.$$

then:
- $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.
- $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges.

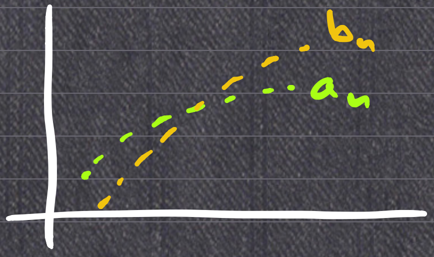
1014 version $\quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1 \Rightarrow \frac{a_{n+1}}{a_n} < \underbrace{\frac{L+1}{2}} \quad \forall n >> 1.$

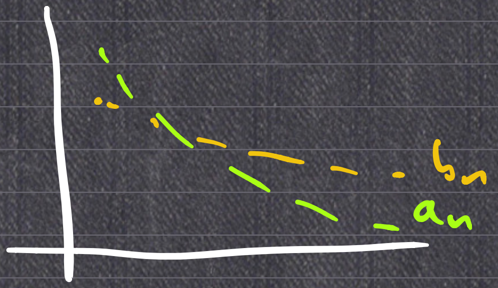
$$= \frac{b_{n+1}}{b_n} \quad \text{where } b_n = \left(\frac{1+L}{2}\right)^n$$

$\frac{a_{n+1}}{a_n}$



$$\frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n}$$





Proof: $\quad \frac{a_{n+1}}{a_n} \le \frac{b_{n+1}}{b_n} \quad \forall n \ge N$

$$a_n = \frac{a_n}{a_{n-1}} \cdot \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} \cdot a_N$$

$$\le \frac{b_n}{\cancel{b_{n-1}}} \cdot \frac{\cancel{b_{n-1}}}{\cancel{b_{n-2}}} \cdots \frac{\cancel{b_{N+1}}}{b_N} \cdot a_N$$

$$\le \underbrace{\frac{a_N}{b_N}}_{\text{constant}} \cdot b_n \qquad \forall n \ge N.$$

$\sum b_n$ converges $\Rightarrow \sum \frac{a_N}{b_N} b_n$ converges

$\underset{\text{comparison test.}}{\Rightarrow} \sum a_n$ converges.

e.g. $\displaystyle\sum_{n=1}^{\infty} \underbrace{\frac{(2n)!}{(n!)^2}\frac{1}{4^n}}_{a_n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{[(n+1)!]^2}\cdot\frac{1}{4^{n+1}}\cdot\frac{4^n\cdot(n!)^2}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2}\cdot\frac{1}{4} = 1.$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{4}\cdot\frac{(2n+2)(2n+1)}{(n+1)^2} = \frac{1}{4}\cdot\frac{2\cancel{(n+1)}(2n+1)}{(n+1)^{\cancel{2}}}$$

$$= \frac{4n+2}{4n+4} = \frac{(4n+4)-2}{4n+4} = 1-\frac{2}{4n+4} = 1-\frac{1}{2(n+1)}$$

$$b_n := \frac{1}{n^p} \Rightarrow \frac{b_{n+1}}{b_n} = \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \left(\frac{n}{n+1}\right)^p$$

$$= \left(1-\frac{1}{n+1}\right)^p = 1-\frac{p}{n+1}+o\left(\frac{1}{n+1}\right)$$

$$(1-x)^p = 1-\underbrace{px}+o(x) \text{ as } x\to 0.$$

$$\left.\frac{d}{dx}\right|_{x=0}(1-x)^p = -p(1-x)^{p-1}\Big|_{x=0} = -p$$

$$\left(1-\frac{1}{n+1}\right)^p = 1-\frac{p}{n+1}+o\left(\frac{1}{n+1}\right) \text{ as } n\to\infty.$$

Hint: compare a_n with $\frac{1}{n^{1/2}}$ (or something close)

$$\frac{a_{n+1}}{a_n} = 1-\frac{1}{2}\cdot\frac{1}{n+1} > 1-\frac{p}{n+1}+o\left(\frac{1}{n+1}\right) \quad \forall n \gg 1.$$

$$= \frac{b_{n+1}}{b_n}$$

if $\frac{1}{2} < p$

$$\Leftarrow \frac{1}{2}\cdot\frac{1}{n+1} < \frac{P}{n+1} + o\left(\frac{1}{n+1}\right) \quad \forall n \gg 1.$$

$$\Leftarrow \left(P-\frac{1}{2}\right)\frac{1}{n+1} + o\left(\frac{1}{n+1}\right) > 0. \quad \forall n \gg 1$$

$$\Leftarrow \frac{1}{n+1}\left(\left(p-\frac{1}{2}\right) + o(1)\right) \underset{\substack{\uparrow \\ \forall n \gg 1}}{>} 0 \Leftarrow p > \frac{1}{2}.$$

Take $p = \frac{3}{4} > \frac{1}{2}$

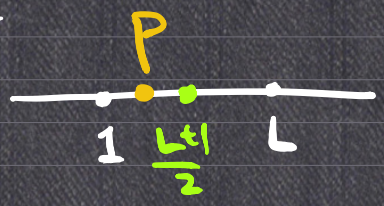
$$\left\{\begin{array}{l} \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}} \text{ diverges.} \\ \frac{a_{n+1}}{a_n} > \frac{b_{n+1}}{b_n} \quad \forall n \gg 1 \end{array}\right. \Longrightarrow \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

Raabe's Test :

Given $\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = L$

if $L > 1$, then $\sum a_n$ converges

if $L < 1$, then $\sum a_n$ diverges.



If $L > 1$,

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right) = L > \frac{L+1}{2}$$

$$\Rightarrow n\left(1-\frac{a_{n+1}}{a_n}\right) > \frac{L+1}{2} \quad \forall n \gg 1.$$

$$\Rightarrow 1-\frac{a_{n+1}}{a_n} > \left(\frac{L+1}{2}\right)\frac{1}{n} \quad \forall n \gg 1$$

$$\Rightarrow \frac{a_{n+1}}{a_n} < 1-\left(\frac{L+1}{2}\right)\frac{1}{n} \quad \forall n \gg 1.$$

$$< 1 - \frac{P}{n} + o\left(\frac{1}{n}\right) \quad \forall n \gg 1.$$

Compare with $b_n = \frac{1}{n^P}$. $(P > 1)$. converges

$\therefore \sum_{n=1}^{\infty} a_n$ converges. $\left(\frac{a_{n+1}}{a_n} < \underbrace{\frac{b_{n+1}}{b_n}}_{= 1 - \frac{p}{n} + o(\frac{1}{n})} \quad \forall n \gg 1 \right).$

$$1 - \frac{p}{n+1} + o\left(\frac{1}{n+1}\right)$$

$$= 1 - p\left(\frac{1}{n} + \underbrace{\frac{1}{n+1} - \frac{1}{n}}_{= \frac{-1}{n(n+1)}} \right) + o\left(\frac{1}{n+1}\right)$$

$$= 1 - \frac{p}{n} + o\left(\frac{1}{n}\right)$$

$\Big\{ \underbrace{\sum_{n=1}^{N} a_n}_{S_N} \Big\}_{N=1}^{\infty}$ increasing if $\underline{a_n \geq 0}$.

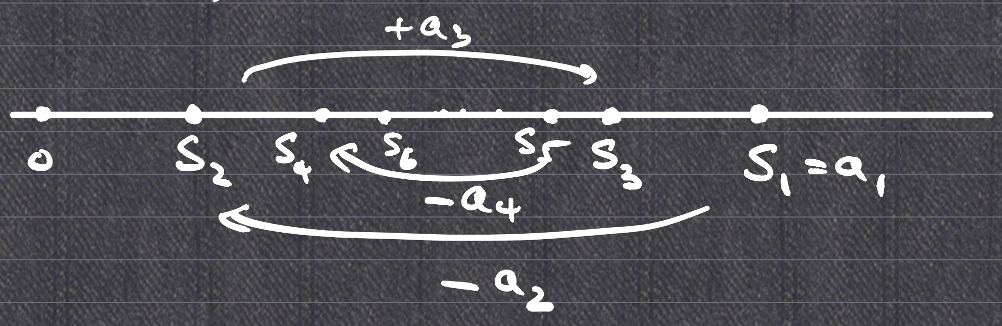
Alternating Series Test:

Given $a_n \geq 0$, $a_n \searrow$, $\lim_{n\to\infty} a_n = 0$.

then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

<u>Proof</u>: $S_n = \sum_{k=1}^{n} (-1)^{k-1} a_k$.

$$S_1 = a_1, \quad S_2 = a_1 - a_2, \quad S_3 = a_1 - a_2 + a_3$$



$$S_{2n} - S_{2n-1} = -a_{2n} \to 0$$

$\square$

e.g. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} = \sum_{n=1}^{\infty} (-1)^n \cdot \underbrace{\frac{1}{n^p}}_{a_n>0}$ converges iff $p>0$.

$\quad\searrow 0$ if $p>0$.

e.g. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cdot \underbrace{\frac{n^2+a}{n^3+b}}_{}$

$a_n \to 0$

let $f(x) := \dfrac{x^2+a}{x^3+b}$, $\quad f'(x) = -\dfrac{x^4-3ax^2+2bx}{(x^3+b)^2}$.

< 0 when $x \gg 1$.

$f(x) \searrow$ when $x \gg 1$.

$\underline{a_n \searrow}$ when $n \gg 1$.

Alt. series test $\Rightarrow \displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ converges.