Integral test:

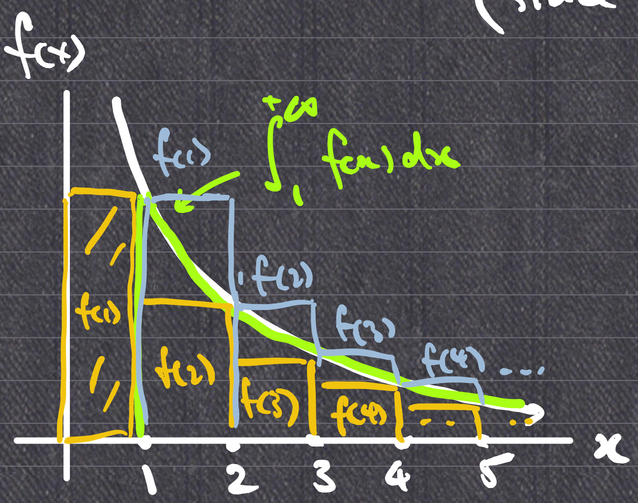
Let $f(x): [1, +\infty) \to \mathbb{R}$ continuous, decreasing

and $\lim\limits_{x \to +\infty} f(x) = 0$. $\longrightarrow$ $f(x) \geq 0$.

$$\sum_{k=1}^{\infty} f(k) \text{ converges} \iff \int_1^{+\infty} f(x)\,dx \text{ converges}$$

e.g. $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

(since $\int_1^{+\infty} \frac{1}{x^p}\,dx$ converges $\iff p > 1$.)



Proof of integral test:

($\Leftarrow$) Suppose $\int_1^{+\infty} f(x)\,dx$ converges.

Want: $\sum\limits_{k=1}^{\infty} f(k)$ converges ($\iff \lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} f(k)$ exists)

$S_n = f(1) + \cdots + f(n)$

$f \geq 0 \Rightarrow S_n$ is increasing

$(S_{n+1} - S_n = f(n+1) \geq 0)$.

On $[k-1, k]$,

$f(x) \geq f(k)$

$\int_{k-1}^{k} f(x) \geq \int_{k-1}^{k} f(k) = f(k)$

$$S_n = f(1) + f(2) + \cdots + f(n)$$
$$\leq f(1) + \int_1^2 f(x)\,dx + \cdots + \int_{n-1}^{n} f(x)\,dx$$
$$= f(1) + \int_1^n f(x)\,dx$$

$$\leq f(1) + \int_1^{+\infty} f(x)\,dx \in \mathbb{R}.$$

$f \geq 0.$

$\therefore S_n$ is bounded above

$\therefore \lim_{n\to\infty} S_n$ exists $\Rightarrow \sum_{k=1}^{\infty} f(k)$ converges

($\Rightarrow$) Given $\sum_{k=1}^{\infty} f(k)$ converges.

Consider $\int_1^{+\infty} f(x)\,dx$:

$F(t) := \int_1^t f(x)\,dx \rightsquigarrow$ want: $\boxed{\lim_{t\to+\infty} F(t) \text{ exists}}$

$F(t)$ increasing because $f \geq 0$.

$$F(t) = \int_1^t f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \dots + \int_{[t]-1}^{[t]} f(x)\,dx + \underbrace{\int_{[t]}^{t} f(x)\,dx}_{\leq \int_{[t]}^{[t]+1} f(x)\,dx}$$

$\int_{k-1}^{k} f(x)dx$

$\leq f(k-1)$

$$\leq f(1) + f(2) + \dots + f([t]-1) + f([t])$$

$$\leq \underbrace{\sum_{k=1}^{\infty} f(k)}_{f(1)+f(2)+\dots+\dots} \in \mathbb{R}.$$

$f(x) \leq f(k-1)$ on $x \in [k-1, k]$

$\Rightarrow \int_{k-1}^{k} f(x)\,dx \leq \int_{k-1}^{k} f(k-1)\,dx$

$= f(k-1)$

$\therefore F(t)$ is bounded above

$\therefore \lim_{t\to+\infty} F(t)$ exists $\Rightarrow \int_1^{+\infty} f(x)\,dx$

converges.

Proposition:

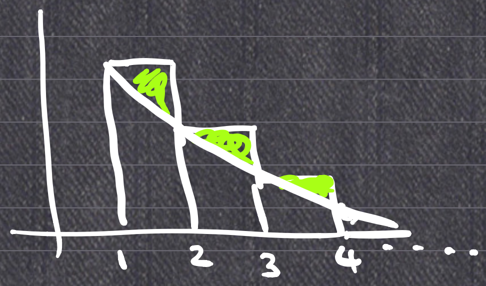
$f: [1, +\infty) \to \mathbb{R}$ continuous, decreasing

$\lim_{x \to +\infty} f(x) = 0.$

then: $\sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx$ (always) converges

Sketch of proof:

$\sigma_n := \sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx$

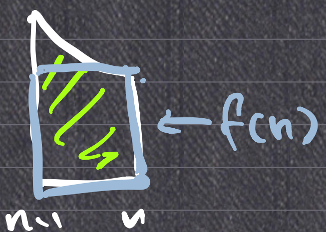


$\sigma_n - \sigma_{n-1}$

$= \sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx$

$- \sum_{k=1}^{n-1} f(k) + \int_1^{n-1} f(x)\,dx$

$= f(n) - \int_{n-1}^{n} f(x)\,dx$



$\leq 0.$

$\therefore \{\sigma_n\}$ is decreasing.

$\sigma_n = \sum_{k=1}^{n} f(k) - \int_1^n f(x)\,dx$

$$= \underbrace{\left(f(1) - \int_1^2 f(x)\,dx\right)}_{\geq 0} + \underbrace{\left(f(2) - \int_2^3 f(x)\,dx\right)}_{\geq 0}$$

$$+ \cdots + \underbrace{\left(f(n-1) - \int_{n-1}^{n} f(x)\,dx\right)}_{\geq 0}$$

$$+ f(n) \geq 0.$$

$f(k-1) \geq \int_{k-1}^{k} f(x)\,dx$

≥ 0.

$\therefore \{\sigma_n\}$ converges.

e.g. $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} - \underbrace{\int_1^n \frac{1}{x}\,dx}$ converges.

$$\gamma := \lim_{n\to+\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} - \log n\right)$$

$\uparrow$ Euler's constant

- comparison test, limit comparison.
- integral test

Next:
- ratio test
- root test.

Ratio test (1014 version):

$\approx$ $\{a_n\}$ is "close" to a geometric series with ratio L.

Given $a_n > 0$, and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} =: L$ exists.

then if $L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

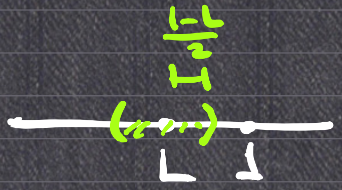
if $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

if $L = 1$, no conclusion.

take 1024.

$a_1, a_2, a_3, a_4, \dots, a_n, a_{n+1}, \dots$

Proof: Case ① $L<1$:



$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$$

$\exists N > 0$ s.t. $n \geq N \Rightarrow \frac{a_{n+1}}{a_n} \in$ (L) 1

$\frac{L+1}{2} < 1.$

$\frac{1-L}{2}$ $\frac{1-L}{2}$

$n \geq N \Rightarrow \frac{a_{n+1}}{a_n} < \frac{L+1}{2}.$

$$a_n < \left(\frac{L+1}{2}\right) a_{n-1} \quad \forall n > N.$$

$$< \left(\frac{L+1}{2}\right)\cdot\left(\frac{L+1}{2}\right) a_{n-2}$$

$$< \cdots < \underbrace{\left(\frac{L+1}{2}\right)^{n-N} a_N}_{} =: b_n$$

$a_{N+1} < \left(\frac{L+1}{2}\right) a_N.$

geometric series

common ratio $= \frac{L+1}{2} < 1.$

$\sum_{n=N}^{\infty} b_n$ converges.

$0 \leq a_n < b_n$ $\Rightarrow$ $\sum_{n=N}^{\infty} a_n$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

comparison test.

Case (2): $L > 1$

1 $\frac{L+1}{2}$ L

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > \frac{L+1}{2}$$

$\Rightarrow \exists N > 0$ s.t. $n \geq N \Rightarrow \frac{a_{n+1}}{a_n} > \underbrace{\frac{L+1}{2}}_{>1}.$

e.g. $\sum_{n=1}^{\infty} \underbrace{\frac{(2n)!}{(n!)^2}}_{a_n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(2n+2)!}{[(n+1)!]^2} \cdot \frac{(n!)^2}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)\cdot\cancel{(2n)!}}{\left[(n+1)\cdot\cancel{n!}\right]^2} \cdot \frac{\cancel{(n!)^2}}{\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)^2}$$

$$= 4 > 1$$

$\therefore \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ diverges.

e.g. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $(p>0)$.

$$\lim_{n\to\infty} \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{1}{\frac{(n+1)^p}{n^p}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^p} = 1.$$

Root test:

Given $a_n \geq 0 \ \forall n$, $r := \lim_{n\to\infty} \sqrt[n]{a_n}$ exists.

then: if $r<1$, then $\sum_{n=1}^{\infty} a_n$ converges.

if $r>1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

if $r=1$, no conclusion. $\longrightarrow$ try something else.

e.g. $\sum_{n=1}^{\infty} \underbrace{(n^5+2n+3)\cdot\frac{1}{2^n}}_{a_n}$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \underline{\sqrt[n]{n^5+2n+3}} \cdot \frac{1}{2}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{n^5+2n+3}{n^5}} \cdot \sqrt[n]{n^5} \cdot \frac{1}{2}$$

$$= \lim_{n\to\infty} \underbrace{\left(1+\frac{2}{n^4}+\frac{3}{n^5}\right)^{\frac{1}{n}}}_{\downarrow\ 1} \cdot \big(\underbrace{n^{\frac{1}{n}}}_{\downarrow\ 1}\big)^5 \cdot \frac{1}{2} = \frac{1}{2} < 1.$$

$$\underset{\downarrow\ 1}{1} \le \left(1+\frac{2}{n^4}+\frac{3}{n^5}\right)^{\frac{1}{n}} \le (1+2+3)^{\frac{1}{n}} = \underset{\downarrow\ 1}{6^{\frac{1}{n}}}$$

$\therefore\ \sum_{n=1}^{\infty} (n^5+2n+3)\cdot\frac{1}{2^n}$ converges.