§ Infinite Series.

$$\sum_{i=1}^{\infty} a_i := \lim_{N\to\infty} \underbrace{\sum_{i=1}^{N} a_i = \lim_{N\to\infty} (a_1 + \cdots + a_N)}_{\text{N-th partial sum.}}$$

$$\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} \sum_{n=0}^{N} ar^n = \lim_{N\to\infty} \frac{a(1-r^{N+1})}{1-r} \qquad r^{N+1} \to 0 \text{ as } N\to\infty$$

$$= \frac{a}{1-r}. \qquad \text{if } |r|<1$$

$$\sum_{n=2}^{\infty} \frac{1}{n(n+1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \lim_{N\to\infty} \sum_{n=2}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \lim_{N\to\infty} \left(\frac{1}{2} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} + \cancel{\cdots} + \cancel{\frac{1}{N}} - \frac{1}{N+1}\right)$$

$$= \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{N+1}\right) = \frac{1}{2}.$$

$$\sum_{n=1}^{\infty} \left(2^{\frac{1}{n}} - 2^{\frac{1}{n+1}}\right) = (2^1 - \cancel{2^{\frac{1}{2}}}) + (\cancel{2^{\frac{1}{2}}} - \cancel{2^{\frac{1}{3}}}) + (\cancel{2^{\frac{1}{3}}} - \cancel{2^{\frac{1}{4}}}) + \cdots$$

$$= 2.$$

$$\sum_{n=1}^{\infty} \left(2^{\frac{1}{n}} - 2^{\frac{1}{n+1}}\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \left(2^{\frac{1}{n}} - 2^{\frac{1}{n+1}}\right)$$

$$= \lim_{N\to\infty} \left(2^1 - \cancel{2^{\frac{1}{2}}} + \cancel{2^{\frac{1}{2}}} - \cancel{2^{\frac{1}{3}}} + \cancel{\cdots} + \cancel{2^{\frac{1}{N}}} - 2^{\frac{1}{N+1}}\right)$$

$$= \lim_{N\to\infty} \left(2 - 2^{\frac{1}{N+1}}\right) = 2 - 1 = 1 \ ✓$$

$$\overset{a_1}{(1-1)}+\overset{a_2}{(1-1)}+\overset{a_3}{(1-1)}+\cdots \leftarrow 0$$

$$= \underset{b_1}{1}-\underbrace{(1-1)}_{b_2}-\underbrace{(1-1)}_{b_3}-\underbrace{(1-1)}_{b_4}-\cdots \leftarrow 1$$

$\cdots$

$a_n = 0 \quad \forall n.$

$\sum_{n=1}^{\infty} a_n = 0.$

$b_1 = 1$

$b_2 = b_3 = b_4 = \cdots = 0.$

$\sum_{n=1}^{\infty} b_n = 1$

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{converges} \quad \underline{s>1}.$$

Top concern: whether an infinite series converges?

Prop: If $\sum_{n=1}^{\infty} a_n$ converges (i.e. $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ exists)

then $\lim_{n\to\infty} a_n = 0$.

(converse is not true

counterexample: $\sum_{n=1}^{\infty} \frac{1}{n}$)

Proof:

$$a_N = \underbrace{\sum_{n=1}^{N} a_n}_{\downarrow} - \underbrace{\sum_{n=1}^{N-1} a_n}_{\downarrow} \quad \text{as } N\to\infty$$

$$\lim_{N\to\infty} a_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} a_n = 0.$$

Cor: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

e.g. $\sum_{n=1}^{\infty} n$, $\sum_{n=1}^{\infty} \frac{n}{n+1}$, $\sum_{n=1}^{\infty} (-1)^n$ all diverge.

$n \to +\infty$, $\frac{n}{n+1} \to 1$, $(-1)^n$ diverges

$\sum_{n=1}^{\infty} n = 1+2+3+4+\cdots = +\infty$

Comparison Test:

Given $\exists m \in \mathbb{N}$ s.t. $b_n \geq a_n \geq 0$ $\forall n \geq m$.

then • $\sum_{n=1}^{\infty} b_n$ converges $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

• $\sum_{n=1}^{\infty} a_n$ diverges $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ diverges.

Proof: $s_N := \sum_{n=1}^{N} a_n$, $t_N := \sum_{n=1}^{N} b_n$ monotone increasing when $N \geq m$.

If $\sum_{n=1}^{\infty} b_n$ converges, $\lim_{N\to\infty} t_n = \sum_{n=1}^{\infty} b_n = \sup\{t_n\}$.

$$\underset{\forall n \geq m}{a_n \leq b_n} \Rightarrow \sum_{n=m}^{N} a_n \leq \sum_{n=m}^{N} b_n \leq \underbrace{\sum_{n=1}^{\infty} b_n}_{\text{finite.}}$$

$$s_N = \sum_{n=1}^{m-1} a_n + \underbrace{\sum_{n=m}^{N} a_n}$$

$$\leq \underbrace{\sum_{n=1}^{m-1} a_n + \sum_{n=1}^{\infty} b_n}_{\text{Constant.}} = (a_1 + a_2 + \cdots + a_{m-1}) + \sum_{n=1}^{\infty} b_n.$$

$\{s_N\}$ is bounded.

$\Rightarrow$ $\lim_{N\to\infty} s_N$ exists $\Rightarrow$ $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N$ converges

$\square$

⚠ If $a_n \ngeq 0$ $\forall$ large n, comparison test cannot be used!

Counterexample: $-\frac{1}{n} \leq \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ converges, but $\sum_{n=1}^{\infty} \left(-\frac{1}{n}\right)$ diverges.

$$-n \le \frac{1}{n^2}$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, but $\sum_{n=1}^{\infty} (-n)$ diverges.

<u>e.g.</u> $$\sum_{n=1}^{\infty} \frac{n+\sin n}{n^3+n+2}$$

$$0 \le \underbrace{\frac{n+\sin n}{n^3+n+2}}_{a_n} \le \frac{n+1}{n^3+n+2} \le \frac{n+1}{n^3} \le \frac{n+n}{n^3} = \underbrace{\frac{2}{n^2}}_{b_n}.$$

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{2}{n^2}$ converges

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

e.g. $$\sum_{n=1}^{\infty} \frac{n+\sin(n)}{n^3-n-2}$$

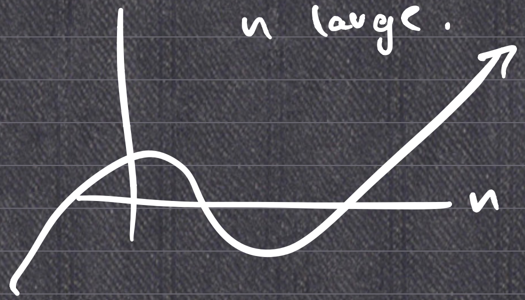
$$n^3 - n - 2 \ge \frac{1}{2} n^3$$

$\Leftrightarrow \frac{1}{2} n^3 - n - 2 \ge 0.$

holds for n large.

$$\frac{n+\sin(n)}{n^3-n-2} \le \frac{n+1}{\frac{1}{2}n^3}$$

$$\le \frac{n+n}{\frac{1}{2}n^3} = \frac{4}{n^2}.$$

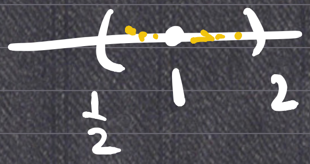


<u>e.g.</u> $$\frac{n+\sin(n)}{n^3-n-2} \approx \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{\frac{n+\sin(n)}{n^3-n-2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2(n+\sin(n))}{n^3-n-2}$$

$$= \lim_{n\to\infty} \frac{n^3+n^2\sin(n)}{n^3-n-2}$$

$$= \lim_{n\to\infty} \frac{1+\underbrace{\frac{\sin n}{n}}_{\to 0}}{1-\frac{1}{n^2}-\frac{2}{n^3}} = 1.$$



for large n :

$$\frac{1}{2} \leq \frac{\frac{n+\sin(n)}{n^3-n-2}}{\frac{1}{n^2}} \leq 2$$

$$\Rightarrow \quad 0 \leq \frac{n+\sin(n)}{n^3-n-2} \leq \frac{2}{n^2} \quad \forall n \text{ large.}$$

Integral Test:

Suppose $f: [1,+\infty) \to [0,\infty)$ is a decreasing function

and $\lim_{x\to+\infty} f(x) = 0$,

then $\sum_{n=1}^{\infty} f(n)$ converges $\Leftrightarrow$ $\int_1^{+\infty} f(x)\,dx$ converges.

as an infinite series — as an improper integral.

e.g. $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ (diverges on $s \leq 0$. how about other s ?)

Suppose $s>0$: $f(x) = \frac{1}{x^s} \searrow$ and $\lim_{x\to+\infty} f(x) = 0.$

$$\zeta(s) = \sum_{n=1}^{\infty} f(n) \quad \text{converges}$$

$$\Longleftrightarrow \underbrace{\int_1^{+\infty} f(x)\,dx}_{\shortparallel} \quad \text{converges.} \quad \Longleftrightarrow \boxed{s > 1}$$

want $-s+1<0$

$$\underbrace{\int_1^{+\infty} \frac{1}{x^s}\,dx}_{\uparrow} = \lim_{\beta\to+\infty} \int_1^{\beta} \frac{1}{x^s}\,dx = \begin{cases} \lim_{\beta\to+\infty} \left[\frac{x^{-s+1}}{-s+1}\right]_1^{\beta} & \text{if } s \neq 1 \\ \lim_{\beta\to+\infty} \left[\log x\right]_1^{\beta} & \text{if } s = 1 \end{cases}$$

converges if $s > 1$

diverges if $s \in (0, 1]$.

e.g. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ $(p > 0)$.

let $f(x) = \dfrac{1}{x(\log x)^p} \searrow$, $\displaystyle\lim_{x\to+\infty} \frac{1}{x(\log x)^p} = 0$.

$$\underbrace{\sum_{n=2}^{\infty} \frac{1}{n(\log x)^p} \text{ converges}} \Longleftrightarrow \underbrace{\int_2^{+\infty} \frac{1}{x(\log x)^p}\,dx}_{\shortparallel} \text{ conv.}$$

$\Longleftrightarrow \boxed{p > 1}$.

$$\lim_{\beta\to+\infty} \int_2^{\beta} \frac{1}{x(\log x)^p}\,dx$$

$$\shortparallel$$

$$\lim_{\beta\to+\infty} \int_2^{\beta} \frac{d(\log x)}{(\log x)^p}$$

$$= \begin{cases} \lim_{\beta\to+\infty} \left[\log(\log x)\right]_2^{\beta} = +\infty & p = 1 \\ \lim_{\beta\to+\infty} \left[\frac{(\log x)^{-p+1}}{1-p}\right]_2^{\beta} & p \neq 1 \end{cases}$$

want $-p+1<0$

$\iff p>1.$