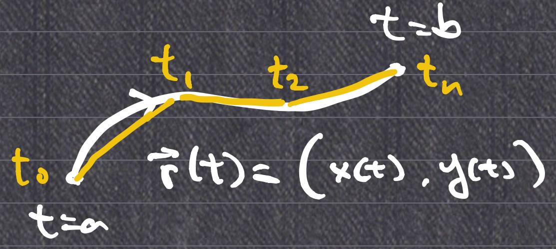
$\vec{r}(t) : [a,b] \to \mathbb{R}^2$ is said to be rectifiable if $\exists C > 0$ s.t. $\forall$ partition $P = \{t_i\}_{i=1}^{n}$ of $[a,b]$,

$$\sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})| \le C$$



then arc-length $:= \sup\left\{ \sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})| : P = \{\overset{a}{\overset{\shortparallel}{t_0}} < t_1 < \cdots < \overset{b}{\overset{\shortparallel}{t_n}}\} \right\}$.

$\leftarrow$ rectifiable

$\pi :=$ arc-length of unit semi-circle.

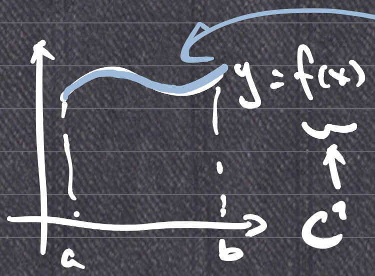
Prop: If $\underbrace{\vec{r}(t)}_{\substack{\shortparallel \\ (f(t),\, g(t))}} : [a,b] \to \mathbb{R}^2$ is C^1, then it is rectifiable.

C^1 on $[a,b]$

Moreover, arc-length $:= \boxed{\int_a^b |\underbrace{\vec{r}'(t)}_{\sqrt{f'(t)^2 + g'(t)^2}}|\, dt}$

$[\vec{r}'(t) = (f'(t), g'(t))]$

Cor:

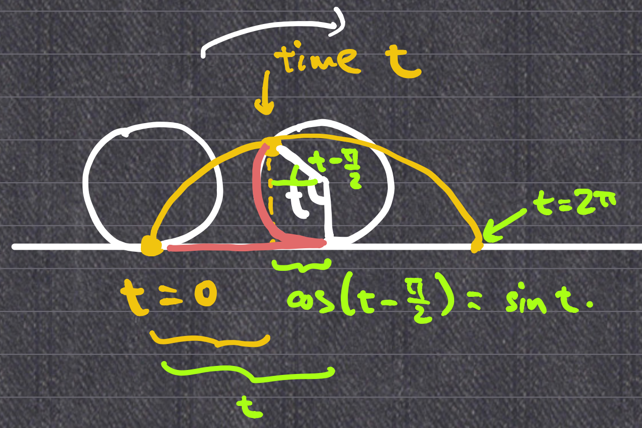


arc-length $= \int_a^b \sqrt{1 + f'(x)^2}\, dx$

$\vec{r}(t) = (t, f(t))$ $\quad \begin{cases} x = t \\ y = f(t) \end{cases}, \quad t \in [a,b]$

$\vec{r}'(t) = (1, f'(t))$

Ex:



$$\begin{cases} x = t - \sin t \\ y = 1 + \sin(t - \frac{\pi}{2}) = 1 - \cos t. \end{cases}$$

$$\vec{r}(t) = (t - \sin t,\ 1 - \cos t).$$

$$\vec{r}\,'(t) = (1 - \cos t,\ \sin t)$$

$$|\vec{r}\,'(t)| = \sqrt{(1-\cos t)^2 + \sin^2 t}$$

$$= \sqrt{2 - 2\cos t}$$

$$\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = \int_0^{2\pi} \sqrt{\sin^2 \tfrac{t}{2}}\, dt$$

$$= \int_0^{2\pi} \underbrace{\sin \tfrac{t}{2}}_{\geq 0}\, dt$$

$$= \left[-2\cos\tfrac{t}{2}\right]_0^{2\pi} = -2(-1-1)$$

$$= 4.$$

e.g. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a, b > 0$.

$$\vec{r}(t) = (a\cos t,\ b\sin t)$$

$$\vec{r}\,'(t) = (-a\sin t,\ b\cos t)$$

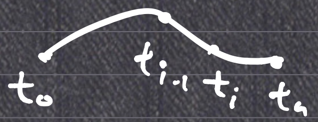
$$|\vec{r}\,'(t)| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$\int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} \sqrt{a^2\sin^2 t + \underbrace{b^2}_{b^2 - a^2 + a^2}\cos^2 t}\ dt$$

$$= \int_0^{2\pi} \sqrt{1 + (b^2 - a^2)\cos^2 t}\ dt$$

Key idea of proof.

$$\int_a^b |\vec{r}'(t)|\,dt \approx \sum_{i=1}^{n} |\vec{r}'(t_i)| \underbrace{\Delta t_i}_{t_i - t_{i-1}}$$



Recall

$$\boxed{|\vec{r}'(t)| \text{ is continuous on } [a,b]}$$

$$\sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})| \approx \sum_{i=1}^{n} |\vec{r}'(\tau_i)(t_i - t_{i-1})|$$

$\tau_i \in [t_{i-1}, t_i]$

If $t_i \approx \tau_i$, then

$$|\vec{r}'(t_i)| \approx |\vec{r}'(\tau_i)|$$

δ

Proof of Prop:

$|\vec{r}'(t)| : [a,b] \to \mathbb{R}$ continuous $\Rightarrow$ Riemann integrable on $[a,b]$

$$\underbrace{\underline{\int_a^b} |\vec{r}'(t)|\,dt}_{\sup_P L(|\vec{r}'|, P)} = \int_a^b |\vec{r}'(t)|\,dt = \underbrace{\overline{\int_a^b} |\vec{r}'(t)|\,dt}_{\inf_P U(|\vec{r}'|, P)}.$$

$\exists \{P_n\}$ partitions of $[a,b]$

s.t.

$$\lim_{n\to\infty} L(|\vec{r}'|, P_n) = \sup_P L(|\vec{r}'|, P)$$

$$= \int_a^b |\vec{r}'(t)|\,dt$$

X $\quad \frac{1}{n}$ $\quad \sup X - \frac{1}{n} < x_n \le \sup X.$

x_n

if $\sup X < +\infty$

$\exists\, x_n \in X$ s.t. $x_n \to \sup X$.

By uniform continuity of f' and g'

$\vec{r}(t) = (f(t), g(t))$

f', g' are C'.

Take $\varepsilon_n = \frac{1}{n}$ $\quad \exists\, \delta_n > 0$ s.t. $|s - t| < \delta_n$

δ_n

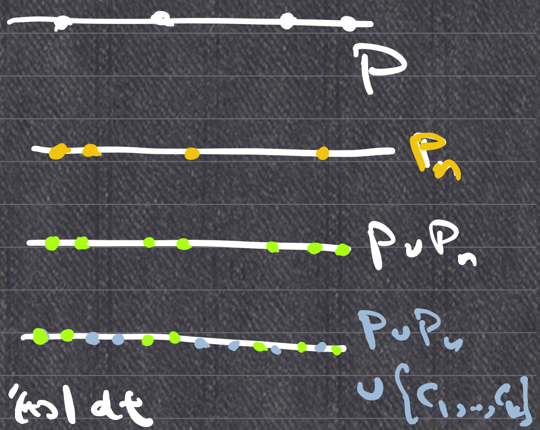
$$\Rightarrow |f'(s) - f'(t)| < \frac{1}{n}$$

and $|g'(s) - g'(t)| < \frac{1}{n}$

Next: prove the rectifiable condition.

Take any arbitrary partition P of $[a,b]$.

$$\left[\underline{\text{want}}: \ l_P := \sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})| \le C. \qquad P = \{t_i\}. \right]$$



$$P_n' := P \cup P_n \cup \{c_1, \dots, c_k\}$$

s.t. Δt_i of $P_n' < \delta_n$.

$$\left\{ \begin{array}{l} l_P \le l_{P_n'} \\[1em] L(|\vec{r}'|, P_n) \le L(|\vec{r}'|, P_n') \le \underline{\int_a^b} |\vec{r}'(t)|\, dt \end{array} \right.$$

$L(|\vec{r}'|, P_n) \to \int_a^b |\vec{r}'(t)|\,dt$, $\quad \underline{\int_a^b} |\vec{r}'(t)|\,dt = \int_a^b |\vec{r}'(t)|\,dt$

$$\boxed{\lim_{n\to\infty} L(|\vec{r}'|, P_n') = \int_a^b |\vec{r}'(t)|\, dt.}$$

let $\underline{P_n' = \{t_i\}_{i=0}^{n}}$

$$\begin{aligned} l_P \le l_{P_n'} &= \sum_{i=1}^{n} |\vec{r}(t_i) - \vec{r}(t_{i-1})| \\ &= \sum_{i=1}^{n} \big| (f(t_i) - f(t_{i-1}),\ g(t_i) - g(t_{i-1})) \big| \\ &= \sum_{i=1}^{n} \big| (f'(t_i^*)(t_i - t_{i-1}),\ g'(t_i^{**})(t_i - t_{i-1})) \big| \\ &= \sum_{i=1}^{n} \big| \underbrace{(f'(t_i^*), g'(t_i^{**}))}_{\vec{v}_i} \big| (t_i - t_{i-1}) \end{aligned}$$

$t_i^*, t_i^{**} \in \underbrace{[t_{i-1}, t_i]}_{< \delta_n}$

$$L(|\vec r\,'|, P_n') = \sum_{i=1}^{n} \inf_{[t_{i-1},t_i]} \underbrace{|\vec r\,'|}_{\text{continuous}} (t_i - t_{i-1})$$

$$= \sum_{i=1}^{n} |\vec r\,'(s_i)| (t_i - t_{i-1})$$

$$= \sum_{i=1}^{n} \Big|\underbrace{(f'(s_i), g'(s_i))}_{\vec w_i}\Big| (t_i - t_{i-1})$$

$$|s_i - t_i^*| < \delta_n \Rightarrow |f'(s_i) - f'(t_i^*)| < \frac{1}{n}$$

$$|s_i - t_i^{**}| < \delta_n \Rightarrow |g'(s_i) - g'(t_i^{**})| < \frac{1}{n}$$

$$\therefore \ |\vec v_i - \vec w_i| = \left(|f'(t_i^*) - f'(s_i)|^2 + |g'(s_i) - g'(t_i^{**})|^2 \right)^{1/2}$$

$$< \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\sqrt{2}}{n}.$$

$$\Rightarrow \ \big||\vec v_i| - |\vec w_i|\big| \le |\vec v_i - \vec w_i| < \frac{\sqrt{2}}{n}.$$

$$\left(\begin{aligned} |\vec v_i| &= |\vec w_i + (\vec v_i - \vec w_i)| \\ &\le |\vec w_i| + |\vec v_i - \vec w_i| \end{aligned} \right) \qquad \begin{aligned} |\vec v_i| - |\vec w_i| &< \frac{\sqrt{2}}{n} \\ |\vec w_i| - |\vec v_i| &< \frac{\sqrt{2}}{n}. \end{aligned}$$

$$l_P \le l_{P_n'} = \sum_{i=1}^{n} |\vec v_i| (t_i - t_{i-1})$$

$$< \sum_{i=1}^{n} \left(|\vec w_i| + \frac{\sqrt{2}}{n} \right) (t_i - t_{i-1})$$

$$= \underbrace{L(|\vec r\,'|, P_n')}_{\downarrow \ \int_a^b |\vec r\,'(t)|\,dt} + \underbrace{\sum_{i=1}^{n} \frac{\sqrt{2}}{n} (t_i - t_{i-1})}_{= \underbrace{\frac{\sqrt{2}}{n}(b-a)}_{\downarrow\ 0}}$$

convergence $\Rightarrow$ bounded.

$$\therefore \quad L(|\vec{r}'|, P_n') + \sum_{i=1}^{n} \frac{\sqrt{2}}{n}(t_i - t_{i-1}) \le C \quad \forall n.$$

$$\Rightarrow \quad l_P \le l_{P_n'} \le \int_a^b |\vec{r}'(t)|\,dt$$

$$\begin{aligned} l_{P_n'} &= \sum_{i=1}^{n} |\vec{v}_i|(t_i - t_{i-1}) \\ &> \sum_{i=1}^{n} \left(|\vec{w}_i| - \frac{\sqrt{2}}{n}\right)(t_i - t_{i-1}) \\ &= L(|\vec{r}'|, P_n') - \frac{\sqrt{2}}{n}(b-a) \end{aligned}$$

$$L(|\vec{r}'|, P_n') - \sum_{i=1}^{n} \frac{\sqrt{2}}{n}(t_i - t_{i-1}) < l_{P_n'} < L(|\vec{r}'|, P_n') + \sum_{i=1}^{n} \frac{\sqrt{2}}{n}(t_i - t_{i-1})$$

$\forall n \in \mathbb{N}$

$$\overset{n\to\infty}{\Rightarrow} \quad \lim_{n\to\infty} l_{P_n'} = \lim_{n\to\infty} L(|\vec{r}'|, P_n') = \int_a^b |\vec{r}'(t)|\,dt.$$

$$\therefore \quad \sup_P l_P = \int_a^b |\vec{r}'(t)|\,dt.$$