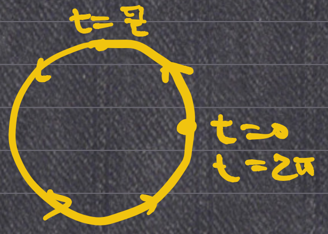
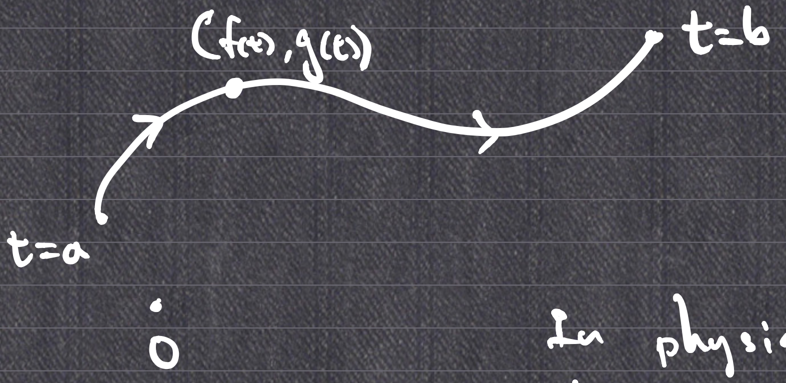
$$\text{circle}: \quad \underbrace{x^2+y^2=1}_{\text{level set}}, \qquad \underbrace{\begin{aligned} x &= \cos t \leftarrow \text{time} \\ y &= \sin t \end{aligned}}_{\text{parametric equations}}$$



generally:

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$$



$$\vec{r}(t) = (f(t), g(t)) = f(t)\,\hat{i} + g(t)\,\hat{j}$$

$t \in [a,b]$

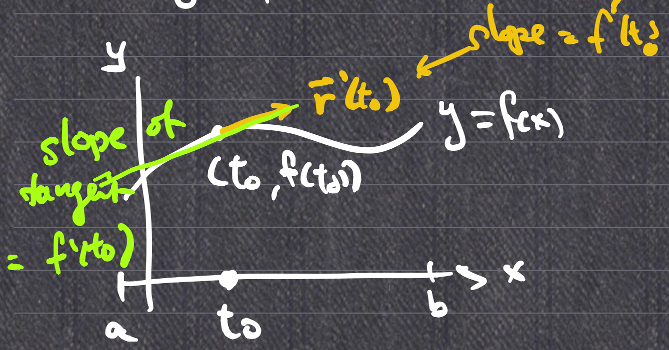
In physics:

velocity: $\vec{r}\,'(t) = (f'(t), g'(t))$

acceleration: $\vec{r}\,''(t) = (f''(t), g''(t))$

Graph of function:

$y = f(x)$

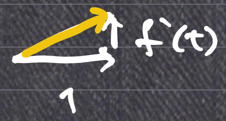


$$\begin{cases} x = t \\ y = f(t) \end{cases}, \quad t \in [a,b]$$

$$\vec{r}(t) = (t, f(t))$$

$$\underbrace{\vec{r}\,'(t)} = (1, f'(t)) = \hat{i} + f'(t)\,\hat{j}$$

slope $= f'(t)$

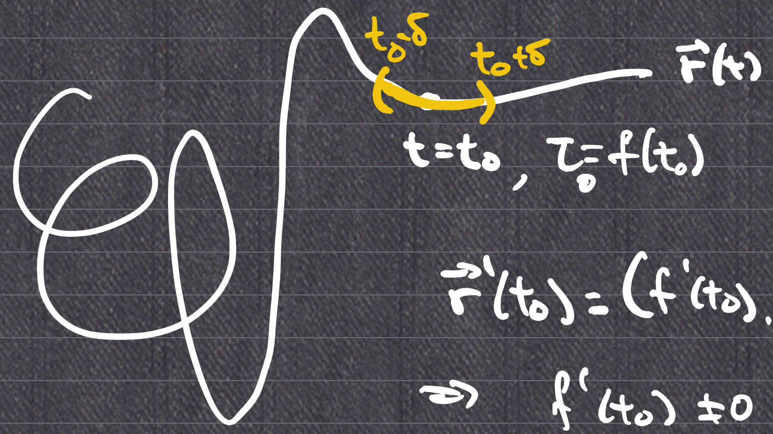


$\therefore \ \vec{r}\,'(t_0) \parallel$ tangent at $(t_0, f(t_0))$

Generally: $\begin{cases} x = f(t) \\ y = g(t) \end{cases}, \quad t \in [a,b]$

Why $\vec{r}\,'(t) \parallel$ tangent line?

At $t=t_0$, assume $\vec{r}\,'(t_0) \neq \vec{0}$ and $\vec{r}\,'(t)$ is C^0.



$t=t_0$, $\tau_0 = f(t_0)$

$(f(t_0), g(t_0))$

$\vec{r}\,'(t_0) = (f'(t_0), g'(t_0)) \neq (0,0)$.

$\Rightarrow \underbrace{f'(t_0) \neq 0}$ or $g'(t_0) \neq 0$.

In case $f'(t_0) \neq 0$, then $\exists \delta > 0$
s.t. $f'(t) \neq 0 \quad \forall t \in (t_0-\delta, t_0+\delta)$

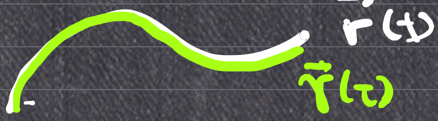
$\Rightarrow f(t)$ is injective on $(t_0-\delta, t_0+\delta)$

$\Rightarrow f^{-1}$ exists near t_0.

$\vec{\gamma}(\tau) := \vec{r}(f^{-1}(\tau)) = (f(f^{-1}(\tau)), g(f^{-1}(\tau)))$

gamma tau

$= (\tau, g\circ f^{-1}(\tau))$ ← graph of $g\circ f^{-1}$.



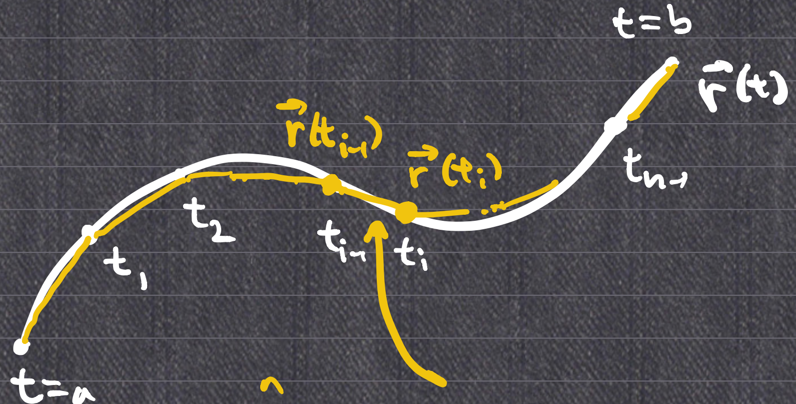
$\underbrace{\vec{\gamma}\,'(\tau_0)}$ // tangent line at $(f(t_0), g(t_0))$

$\shortparallel$

$\vec{r}\,'(\underbrace{f^{-1}(\tau_0)}) \cdot \underbrace{\frac{d}{d\tau} f^{-1}(\tau)\Big|_{\tau_0}}_{\text{scalar}}$ // $\vec{r}\,'(t_0)$

$\tau_0 = f(t_0)$

Arc length?



$P: \{t_0 < t_1 < \cdots < t_n\}$

$$l_P := \sum_{i=1}^{n} |\vec r(t_{i-1}) - \vec r(t_i)|$$

<u>Definition:</u> $\vec r(t)$, $t \in [a,b]$ is <u>rectifiable</u> if $\exists\, C > 0$ constant such that

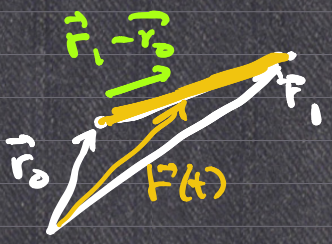
$$l_P \le C \quad \forall \text{ partition } P \text{ of } [a,b].$$

then arc length $:= \sup_P l_P$

$= \sup\{l_P : P \text{ is partition of } [a,b]\}$.

<u>e.g.</u> $\vec r(t) = \vec r_0 + t(\vec r_1 - \vec r_0)$

$t \in [0,1]$



$P: \underset{0}{\overset{\shortparallel}{t_0}} < t_1 < \cdots < t_{n-1} < \underset{1}{\overset{\shortparallel}{t_n}}$

$$l_P = \sum_{i=1}^{n} |\vec r(t_i) - \vec r(t_{i-1})| = \sum_{i=1}^{n} \left| \cancel{\vec r_0} + t_i(\vec r_1 - \vec r_0) - \cancel{\vec r_0} - t_{i-1}(\vec r_1 - \vec r_0) \right|$$

$$= \sum_{i=1}^{n} |\vec r_1 - \vec r_0|(t_i - t_{i-1}) = (t_n - t_0)|\vec r_1 - \vec r_0|$$

$$= \underbrace{|\vec r_1 - \vec r_0|}_{C}$$

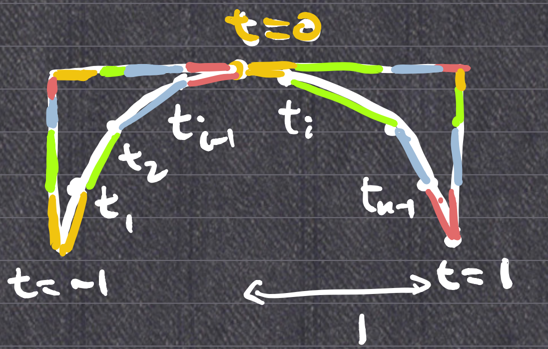
$\underset{C}{\overset{|\vec r_1 - \vec r_0|}{\shortparallel}}$

$\therefore$  is rectifiable. Arc length $= \sup_P l_P = |\vec r_1 - \vec r_0|$.

e.g. a unit circle $x^2+y^2=1$.

$$\vec{r}(t) := \left(t, \sqrt{1-t^2}\right), \quad t \in [-1,1].$$

$$\left.\begin{array}{l} P : -1 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1. \\ P' := P \cup \{0\} \end{array}\right\} \Rightarrow l_P \le l_{P'}$$



$$l_P \le l_{P'} \le 4 \quad \forall P.$$

$$|\vec{r}(t_{i-1}) - \vec{r}(t_i)|$$
$$\le |\vec{r}(t_{i-1}) - \vec{r}(0)| + |\vec{r}(0) - \vec{r}(t_i)|$$

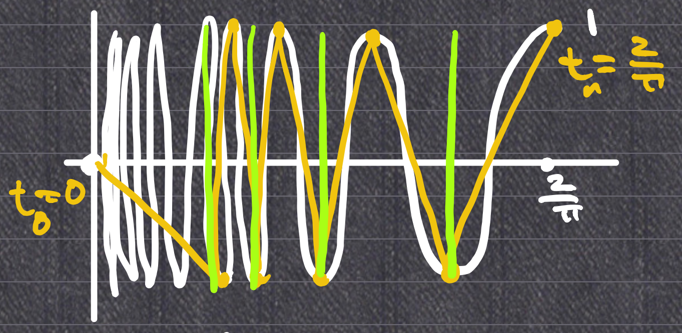
$\therefore$ a semi-circle is rectifiable ($x^2+y^2=1$, $y \ge 0$).

$$\left(\pi := \text{arc length} = \sup_P l_P\right)$$

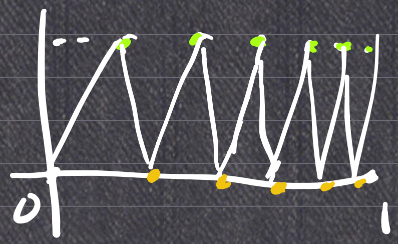
$$\vec{r}(t) = (\cos t, \sin t), \quad t \in [0, 2\pi].$$

$$\vec{r}(t) := \begin{cases} (0,0) & \text{if } t = 0 \\ \left(t, \sin\frac{1}{t}\right) & \text{if } t \in \left(0, \frac{2}{\pi}\right] \end{cases}$$

topologist's sine curve.



$\exists \{P_n\}$ s.t. $l_{P_n} \to +\infty \longrightarrow$ not rectifiable.



$\vec{r}(t) = (t, \mathbb{1}_{\mathbb{Q}}(t))$

$t \in [0,1]$.