4.8 Parametric Curves

In $\mathbb{R}^2$ (similarly for $\mathbb{R}^3$), we often represent a curve in parametric form, such as the unit circle:

$$\begin{aligned} x &= \cos t \\ y &= \sin t \end{aligned}$$

where $t \in [0, 2\pi]$. One may denote these parametric equations in vector form:

$$\mathbf{r}(t) = (\cos t, \sin t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j}, \quad t \in [0, 2\pi].$$

The graph $y = f(x)$ of a single variable function $f : [a, b] \to \mathbb{R}$ can be represented in parametric form by:

$$\mathbf{r}(t) = (t, f(t)), \quad t \in [a, b].$$

It is helpful to think of t as the *time* variable, and $\mathbf{r}(t)$ as the *position vector* of a moving particle at time t. Then the curve represented by $\mathbf{r}(t)$ is the path of the particle.

4.8.1 Geometric meaning of $\mathbf{r}'(t)$

In physics, the meaning of $\mathbf{r}'(t)$ is defined to be the velocity at time t as it is the rate of change of the position vector, and $\mathbf{r}''(t)$ is the acceleration. In mathematics, one can show that $\mathbf{r}'(t)$ (if non-zero) is in fact a tangent vector to the curve $\mathbf{r}(t)$.

To prove this, we first show that it is true when the curve is a graph of a C^1 function f, i.e. the special case $\mathbf{r}(t) = (t, f(t))$. Then, we have

$$\mathbf{r}'(t) = (1, f'(t)) = \mathbf{i} + f'(t)\mathbf{j}.$$

This vector has slope $f'(t)$, which is the slope of the tangent to the graph $y = f(x)$ at the point $(t, f(t))$. Hence, $\mathbf{r}'(t)$ is a tangent vector to the curve at the point $(t, f(t))$.

Now consider a general case $\mathbf{r}(t) = (f(t), g(t))$ where f, g are C^1 functions. If $\mathbf{r}'(t_0) \neq \mathbf{0}$ at a particular time t_0, then $f'(t_0) \neq 0$ or $g'(t_0) \neq 0$. WLOG we assume $f'(t_0) \neq 0$. Then, f is strictly monotone near t_0, and hence it is locally invertible. From MATH 1023, we learned that the local inverse $f^{-1} : (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \to (t_0 - \delta, t_0 + \delta)$ is also C^1. Here $f(t_0) = \tau_0$. Next we consider a "new" curve:

$$\gamma(\tau) := \mathbf{r}(f^{-1}(\tau)) = \big(\tau, g \circ f^{-1}(\tau)\big), \quad \tau \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon).$$

We put "new" in quotation because it is not really a new curve, but the same curve as $\mathbf{r}(t)$ near $t = t_0$ with the particle travelling at a different speed. By now, the curve $\gamma(\tau)$ is simply the graph of $y = g \circ f^{-1}(x)$. From the previous paragraph, we know that $\gamma'(\tau_0)$ is a tangent vector to the curve at the point $\gamma(\tau_0) = (f(t_0), g(t_0))$. However, by chain rule, we also know that

$$\gamma'(\tau) = \frac{d}{d\tau}\mathbf{r}(f^{-1}(\tau)) = \mathbf{r}'(f^{-1}(\tau))\frac{d}{d\tau}f^{-1}(\tau) \implies \gamma'(\tau_0) = \underbrace{\frac{d}{d\tau}f^{-1}(\tau)\bigg|_{\tau=\tau_0}}_{\text{scalar}} \mathbf{r}'(t_0).$$

Therefore, $\gamma'(\tau_0)$ and $\mathbf{r}'(t_0)$ are parallel to each other, and so $\mathbf{r}'(t_0)$ is also a tangent vector to the curve at $(f(t_0), g(t_0))$. The case when $g'(t_0) \neq 0$ is similar – just regard x is a function y near the point $(f(t_0), g(t_0))$.

> The geometric meaning of $\mathbf{r}''(t)$ is related to the curvature. You may learn more about it in MATH 2023 or 4223.

4.8.2 Rectifiable Curves

Next we discuss what it means by *length* of a curve. Given a curve $\mathbf{r}(t) : [a, b] \to \mathbb{R}^2$, we first attempt to approximate it by line segments. That is, take a partition $P = \{a = t_0 < t_1 < \cdots < t_n = b\}$ and consider the sum:

$$l_P := \sum_{i=1}^{n} |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|.$$

It is the total length of the line segments joining points $\mathbf{r}(t_0), \mathbf{r}(t_1), \cdots, \mathbf{r}(t_n)$. As we are taking more and more refined partitions P, we expect l_P gets larger by the triangle inequality. Therefore, the *best* approximation of the length of the curve is naturally defined as the maximum possible l_P among all partitions P.

Definition 4.7 — Rectifiable Curve and Arc Length. Let $\mathbf{r}(t) : [a,b] \to \mathbb{R}^2$ be a curve in $\mathbb{R}^2$. We call $\mathbf{r}(t)$ a **rectifiable curve** if $l_P \leq C$ for some constant $C \in (0,\infty)$ independent of partitions P of $[a,b]$. In such case, we define the **arc length** of $\{\mathbf{r}(t)\}_{t\in[a,b]}$ to be:

$$\sup_P l_P = \sup\left\{\sum_{i=1}^n |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| : a = t_0 < t_1 < \cdots < t_n = b\right\}.$$

It doesn't seem easy to check whether a curve is rectifiable nor to compute the arc length. Fortunately, we can later show that any C^1 curve (i.e. $\mathbf{r}(t) = (f(t), g(t))$ where f, g are C^1) is rectifiable and its arc length is simply given by the integral $\displaystyle\int_a^b |\mathbf{r}'(t)|\, dt$.

Obviously, any straight-line segment $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, $t \in [0,1]$, joining the points with position vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ is rectifiable. To prove this, we compute that for any partition $P : 0 = t_0 < t_1 < \cdots < t_n = 1$, we have:

$$\sum_{i=1}^n |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})| = \sum_{i=1}^n |(t_i - t_{i-1})\mathbf{r_1} - \mathbf{r_0}| = \sum_{i=1}^n (t_i - t_{i-1})\,|\mathbf{r}_1 - \mathbf{r}_0| = |\mathbf{r}_1 - \mathbf{r}_0|\,.$$

In particular, $l_P = |\mathbf{r}_1 - \mathbf{r}_0|$ for any partition P of $[0,1]$, hence it is bounded above. This shows the straight-line segment is rectifiable, and its length is given by:

$$\sup_P l_P = \sup_P |\mathbf{r}_1 - \mathbf{r}_0| = |\mathbf{r}_1 - \mathbf{r}_0|\,,$$

which is exactly what we expect.

We next show that a unit circle is rectifiable. We want to avoid using any differentiation on $\sin$ and $\cos$ functions, because they are based on the limit identity $\frac{\sin x}{x} \to 1$ when $x \to 0$. The proof of this limit identity requires the use of length of a circular arc, so it is built upon the fact that a circle is rectifiable. To prove that a unit circle is rectifiable without circular reasoning, we parametrize the semi-circle by:

$$\mathbf{r}(t) := \left(t, \sqrt{1-t^2}\right),\ t \in [-1,1].$$

After showing also the lower semi-circle is also rectifiable (*mutatis mutandis*), then we can conclude that the full circle is rectifiable. We need the following observation:

■ **Exercise 4.77** Let P be a partition of $[a,b]$, and let $P' = P \cup \{t'\}$. Show that $l_P \leq l_{P'}$. Hence, show that for a continuous curve $\{\mathbf{r}(t)\}_{t\in[a,b]}$, if $\{\mathbf{r}(t)\}_{t\in[a,c]}$ and $\{\mathbf{r}(t)\}_{t\in[c,b]}$ are rectifiable for some $c \in (a,b)$, then $\{\mathbf{r}(t)\}_{t\in[a,b]}$ is also rectifiable.

Recall that we parametrize the upper semi-circle by $\mathbf{r}(t) := \left(t, \sqrt{1-t^2}\right)$, $t \in [-1,1]$. Given any partition P of $[-1,1]$, we may refine P by taking $P' := P \cup \{0\}$, then we must have $l_P \leq l_{P'}$. After such a refinement, one can easily see from the diagram below that $l_{P'} \leq 4$:

Figure 4.6: diagram to be added

In particular, we have $l_P \leq l_{P'} \leq 4$ for any partition P of $[-1,1]$. This shows the upper semi-circle is rectifiable, meaning that $\sup_P l_P$ exists in $\mathbb{R}$. We then define

$$\pi := \sup_P l_P.$$

■ **Exercise 4.78** Show that if $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a distance-preserving map, and $\{\mathbf{r}(t)\}_{t\in[a,b]}$ is a rectifiable curve, then $\{\Phi \circ \mathbf{r}(t)\}_{t\in[a,b]}$ is also rectifiable and its length is the same as that of $\{\mathbf{r}(t)\}_{t\in[a,b]}$.

4.8.3 An example of a non-rectifiable curve

A curve could be non-rectifiable if it fluctuates too much, such as:

$$\mathbf{r}(t) = \begin{cases} (0,0) & \text{if } t = 0 \\ (t, \sin\frac{1}{t}) & \text{if } 0 < t \le \frac{2}{\pi} \end{cases}.$$

To see why it is not rectifiable, we consider the sequence of partitions:

$$P_n : 0 < \frac{1}{\frac{\pi}{2} + 2n\pi} < \frac{1}{-\frac{\pi}{2} + 2n\pi} < \frac{1}{\frac{\pi}{2} + 2(n-1)\pi} < \frac{1}{-\frac{\pi}{2} + 2(n-1)\pi} < \cdots < \frac{1}{-\frac{\pi}{2} + 2\pi} < \frac{1}{\frac{\pi}{2}}.$$

Then, we can easily see that

$$\begin{aligned} l_{P_n} &\ge \sum_{k=1}^{n} \left| \mathbf{r}\left(\frac{1}{\frac{\pi}{2} + 2k\pi}\right) - \mathbf{r}\left(\frac{1}{-\frac{\pi}{2} + 2k\pi}\right) \right| \\ &\ge \sum_{k=1}^{n} \left| \sin\left(\frac{\pi}{2} + 2k\pi\right) - \sin\left(-\frac{\pi}{2} + 2k\pi\right) \right| \\ &\ge \sum_{k=1}^{n} 2 = 2n. \end{aligned}$$

Since $n \in \mathbb{N}$ can be arbitrarily large, it is impossible to find an upper bound C for l_{P_n}. This concludes such the curve $\mathbf{r}(t)$ is not rectifiable.

■ **Exercise 4.79** Show that the graph $y = f(x)$, $x \in [0, 2/\pi]$ where

$$f(x) = \begin{cases} x \sin\frac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not rectifiable.

4.8.4 Arc-length formula for C^1 curves

Now we are ready to derive the formula of arc-length that appears in many calculus textbooks for physics/engineering majors.

Proposition 4.30 Suppose $\mathbf{r}(t) = (f(t), g(t))$, $t \in [a,b]$, is a curve with f, g being C^1 on $[a,b]$. Then, $\{\mathbf{r}(t)\}_{t=[a,b]}$ is rectifiable, and its arc-length is given by:

$$\int_a^b |\mathbf{r}'(t)|\,dt = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt.$$

Proof. The key idea is the use the mean value theorem to relate $\mathbf{r}'(t)$ and $\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})$. First note that $|\mathbf{r}'(t)|$ is a continuous function on $[a,b]$, so it is Riemann integrable on $[a,b]$. Hence, we have

$$\sup_P L(|\mathbf{r}'|, P) = \underline{\int_a^b} |\mathbf{r}'(t)|\,dt = \int_a^b |\mathbf{r}'(t)|\,dt = \overline{\int_a^b} |\mathbf{r}'(t)|\,dt = \inf_P U(|\mathbf{r}'|, P).$$

By the standard $\frac{1}{n}$-trick, one can take a sequence of partitions $\{P_n\}$ of $[a,b]$ such that

$$\lim_{n\to\infty} L(|\mathbf{r}'|, P_n) = \sup_P L(|\mathbf{r}'|, P).$$

Since $f'(t)$ and $g'(t)$ are continuous on the closed and bounded interval $[a,b]$, they are also uniformly continuous on $[a,b]$. Hence, for any $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that whenever $|t-s| < \delta_n$, we have $|f'(t) - f'(s)| < \frac{1}{n}$ and $|g'(t) - g'(s)| < \frac{1}{n}$.

Now given any partition P of $[a,b]$, we need to bound l_P by a constant, and that $\int_a^b |\mathbf{r}'(t)|\ dt$ is the least upper bound of l_P's among all partitions P of $[a,b]$. By mixing P with P_n, and with enough partition points $\{c_1, \cdots, c_k\}$, we can get a refined partition $P'_n = P \cup P_n \cup \{c_1, \cdots, c_k\}$ such that all its subintervals have width less than δ_n. By reordering the partition points, we denote:

$$P'_n = \{a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b\}$$

Then, we have $t_i - t_{i-1} < \delta_n$ for any i. Then, we still have

$$\lim_{n\to\infty} L(|\mathbf{r}'|, P'_n) = \int_a^b |\mathbf{r}'(t)|\ dt$$

since

$$L(|\mathbf{r}'|, P_n) \le L(|\mathbf{r}'|, P'_n) \le \int_a^b |\mathbf{r}'(t)|\ dt.$$

Next on each $[t_{i-1}, t_i]$, we use the mean value theorem compare $|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|$ with the term $\inf_{[t_{i-1},t_i]} |\mathbf{r}'| (t_i - t_{i-1})$ in $L(|\mathbf{r}'|, P'_n)$. By extreme value theorem and continuity of $|\mathbf{r}'(t)|$, there exists $s_i \in [t_{i-1}, t_i]$ such that

$$\inf_{[t_{i-1},t]} |\mathbf{r}'| = |\mathbf{r}'(s_i)|.$$

Also, mean value theorem shows there exists $t_i^*, t_i^{**} \in (t_{i-1}, t_i)$ such that

$$\begin{aligned} f(t_i) - f(t_{i-1}) &= f'(t_i^*)(t_i - t_{i-1}),\\ g(t_i) - g(t_{i-1}) &= g'(t_i^{**})(t_i - t_{i-1}). \end{aligned}$$

Then, we have

$$l_P \le l_{P'_n} = \sum_{i=1}^n |\mathbf{r}'(t_i) - \mathbf{r}'(t_{i-1})| = \sum_{i=1}^n \left|\left(f'(t_i^*), g'(t_i^{**})\right)\right| (t_i - t_{i-1}).$$

Note that by $s_i, t_i^*, t_i^{**} \in [t_{i-1}, t_i]$ where $t_i - t_{i-1} < \delta_n$, so we have

$$|f'(t_i^*) - f'(s_i)| < \frac{1}{n} \quad \text{and} \quad |g'(t_i^{**}) - g'(s_i)| < \frac{1}{n}.$$

This shows

$$\left|\left(f'(t_i^*), g'(t_i^{**})\right) - \left(f'(s_i), g'(s_i)\right)\right| = \sqrt{|f'(t_i^*) - f'(s_i)|^2 + |g'(t_i^{**}) - g'(s_i)|^2} < \frac{\sqrt{2}}{n}.$$

Then by the (corollary of) triangle inequality in $\mathbb{R}^2$: $|\mathbf{v}| \le |\mathbf{v} - \mathbf{w}| + |\mathbf{w}|$ for any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, we have

$$\begin{aligned} \left|\left(f'(t_i^*), g'(t_i^{**})\right)\right| &\le \left|\left(f'(t_i^*), g'(t_i^{**})\right) - \left(f'(s_i), g'(s_i)\right)\right| + \left|\left(f'(s_i), g'(s_i)\right)\right| \\ &< \frac{\sqrt{2}}{n} + |\mathbf{r}'(s_i)| = \frac{\sqrt{2}}{n} + \inf_{[t_{i-1},t_i]} |\mathbf{r}'|. \end{aligned}$$

Therefore, we conclude that:

$$\begin{aligned} l_P \le l_{P'_n} &= \sum_{i=1}^n \left|\left(f'(t_i^*), g'(t_i^{**})\right)\right| (t_i - t_{i-1}) \\ &< \left(\frac{\sqrt{2}}{n} + \inf_{[t_{i-1},t_i]} |\mathbf{r}'|\right)(t_i - t_{i-1}) \\ &= \frac{\sqrt{2}}{n}(b-a) + L(|\mathbf{r}'|, P'_n). \end{aligned}$$

Letting $n \to \infty$, we proved that

$$l_P \leq \lim_{n\to\infty} \left(\frac{\sqrt{2}}{n}(b-a) + L(|\mathbf{r}'|, P_n') \right) = \int_a^b |\mathbf{r}'(t)| \, dt.$$

Therefore, l_P is bounded from above by a constant $\int_a^b |\mathbf{r}'(t)| \, dt$ independent of t. This shows the curve $\{\mathbf{r}(t)\}_{t\in[a,b]}$ is rectifiable.

To show that $\sup_P l_P = \int_a^b |\mathbf{r}'(t)| \, dt$, we first show that

$$\lim_{n\to\infty} l_{P_n'} = \int_a^b |\mathbf{r}'(t)| \, dt.$$

Recall that

$$\left| \left(f'(t_i^*), g'(t_i^{**})\right) - \left(f'(s_i), g'(s_i)\right) \right| < \frac{\sqrt{2}}{n}.$$

By $|\mathbf{w}| \geq -|\mathbf{v} - \mathbf{w}| + |\mathbf{v}|$, we have

$$\begin{aligned} \left| \left(f'(t_i^*), g'(t_i^{**})\right) \right| &\geq -\left| \left(f'(t_i^*), g'(t_i^{**})\right) - \left(f'(s_i), g'(s_i)\right) \right| + \left| \left(f'(s_i), g'(s_i)\right) \right| \\ &> -\frac{\sqrt{2}}{n} + |\mathbf{r}'(s_i)| = -\frac{\sqrt{2}}{n} + \inf_{[t_{i-1}, t_i]} |\mathbf{r}'|. \end{aligned}$$

This shows

$$l_{P_n'} = \sum_{i=1}^n \left| \left(f'(t_i^*), g'(t_i^{**})\right) \right| (t_i - t_{i-1}) > -\frac{\sqrt{2}}{n}(b-a) + L(|\mathbf{r}'|, P_n').$$

Combining with earlier result, we have

$$-\frac{\sqrt{2}}{n}(b-a) + L(|\mathbf{r}'|, P_n') < l_{P_n'} < +\frac{\sqrt{2}}{n}(b-a) + L(|\mathbf{r}'|, P_n').$$

Letting $n \to \infty$ and by squeeze theorem, we proved:

$$\lim_{n\to\infty} l_{P_n'} = \int_a^b |\mathbf{r}'(t)| \, dt$$

- $\int_a^b |\mathbf{r}'(t)| \, dt$ is an upper bound of l_P over all partitions P of $[a, b]$, and
- there exists a sequence $\{P_n'\}$ such that

$$\lim_{n\to\infty} l_{P_n'} = \int_a^b |\mathbf{r}'(t)| \, dt.$$

These combined show that $\sup_P l_P = \int_a^b |\mathbf{r}'(t)| \, dt$. It proves that this integral gives the length of the curve. ■

■ **Exercise 4.80** Show that if L is an upper bounded of X, and there exists a sequence $x_n \in X$ such that $\lim_{n\to\infty} x_n = L$, then we have $\sup X = L$.

■ **Exercise 4.81** First digest the whole proof of Proposition 4.30. In the proof we considered $\sup_P L(|\mathbf{r}'|, P)$ to extract a sequence $\{P_n\}$ so that $L(|\mathbf{r}'|, P_n)$ converges to $\int_a^b |\mathbf{r}'(t)| \, dt$. Can we prove the proposition by considering $\inf_P U(|\mathbf{r}'|, P)$ instead? If not, point out why. If yes, rewrite the whole proof (without looking at the above proof) by considering $\inf_P U(|\mathbf{r}'|, P)$.

Using the arc-length formula, one can easily derive that the length of the graph $y = f(x)$ over $x \in [a, b]$ is given by:

$$\int_a^b \sqrt{1 + f'(x)^2}\, dx$$

It is simply because we can parametrize the graph by $\mathbf{r}(t) = (t, f(t))$, $t \in [a, b]$. One can check easily that $|\mathbf{r}'(t)| = |(1, f'(t))| = \sqrt{1 + |f'(t)|^2}$.

■ **Exercise 4.82** When you ride a bicycle near a farm field and a piece of cow's dung sticks on your wheel. The trajectory of the dung is given by:

$$\mathbf{r}(t) = (rt - r\sin t, r - r\cos t),$$

where $r > 0$ is the radius of the wheel (assuming r is much larger than the diameter of the dung). Find the distance travelled by the dung after one cycle.

■ **Exercise 4.83** Write down a parametrization $\mathbf{r}(t)$ of the curve $x^{2/3} + y^{2/3} = 1$, and compute its arc length.

■ **Exercise 4.84** A polar curve is one that is given by an equation $r = f(\theta)$, $\theta \in [\alpha, \beta]$. Here (r, θ) denote the polar coordinates and f is a C^1 function of θ. Show that the length of the curve is given by

$$\int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta.$$