$f \in C^0,\ g \in C^1$ on $[a, +\infty)$ $\qquad F(x) := \int_a^x f(t)\,dt$

Dirichlet Test : $\begin{cases} F(x) \text{ bounded} \\ g \text{ monotone and } \lim_{x \to +\infty} g(x) = 0. \end{cases}$

Abel Test : $\begin{cases} \lim_{x \to +\infty} F(x) \text{ exists}, \\ g \text{ monotone, bounded.} \end{cases}$

$$\Rightarrow \underbrace{\int_a^{+\infty} f(x) g(x)\,dx}_{\shortparallel} \text{ converges}$$

$$\int_a^{+\infty} g(x)\,dF(x) = \Big[g(x) F(x)\Big]_a^{+\infty} - \int_a^{+\infty} F(x) g'(x)\,dx$$

$$\int_a^{+\infty} |F(x) g'(x)|\,dx$$

$$\leq \int_a^{+\infty} M\, g'(x)\,dx$$

$$= \lim_{b \to +\infty} M(g(b) - g(a))$$

$$\underset{\uparrow}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{1}{n^s} \text{ converges iff } \underline{s > 1}.$$

$(\mathrm{Re}(s) > 1)$.

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t}\,dt$$

$$= \underbrace{\int_0^1 t^{s-1} e^{-t}\,dt}_{n} + \int_1^{+\infty} \underbrace{t^{s-1} e^{-t}}_{\leq e^{-t/2}}\,dt$$

$f(x) := \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ when $|x| < 1$.

converges $\left(\frac{1}{t^{1-s}}\right)$
if $s-1 > -1$
$\Leftrightarrow s > 0.$

$f: (-1,1) \to \mathbb{R}.$

$\hat{f}(x) := \frac{x}{1-x} : \mathbb{R}\setminus\{1\} \to \mathbb{R}.$

$\hat{f} = f$ on $(-1,1).$

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$$

let $\tau = \frac{t}{n}$ $(n \in \mathbb{N})$

$d\tau = \frac{1}{n} dt,$ when $t = 0$, $\tau = 0$
when $t = +\infty$, $\tau = +\infty$

$$\Gamma(s) = \int_0^{+\infty} (\underbrace{n\tau}_{t})^{s-1} e^{-n\tau} \cdot n\, d\tau$$

$$= \int_0^{+\infty} \underbrace{n^s}\, t^{s-1} e^{-nt} dt.$$

$$\frac{1}{n^s}\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-nt} dt.$$

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^s}\Gamma(s)} = \sum_{n=1}^{\infty} \int_0^{+\infty} t^{s-1} e^{-nt} dt$$

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty} \int_0^{+\infty} t^{s-1} e^{-nt} dt$$

$$= \int_0^{+\infty} \sum_{n=1}^{\infty} t^{s-1} (e^{-t})^n dt \qquad (3033, 3043)$$

$$= \int_0^{+\infty} t^{s-1} \cdot \frac{e^{-t}}{1-e^{-t}} dt = \int_0^{+\infty} t^{s-1} \cdot \frac{1}{e^t - 1} dt$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{e^t - 1} dt \qquad (s > 1)$$

Exercise: $\int_0^{+\infty} t^{s-1} \cdot \frac{1}{e^t-1}\, dt$ converges when $\left(\begin{array}{c} s>0 \\ \text{and } s \neq 1 \end{array}\right)$

$$\hat{\zeta}(s) : (0,+\infty)\setminus\{1\} \to \mathbb{R}$$

$$\hat{\zeta}(s) := \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{e^t-1}\, dt$$

$$\hat{\zeta}(s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{when } s>1.$$