Tutorial 7 sol

1(a) $\int_{-\infty}^{\infty} \frac{1}{x^3-1}\, dx = \int_{-\infty}^{\infty} \frac{1}{(x-1)(x^2+x+1)}$

$\int_1^2 \frac{1}{x^3-1} \geq \int_1^2 \frac{1}{(x-1)(2^2+2+1)} = c\int_1^2 \frac{1}{x-1} = \infty$

(b) $\int_{-\infty}^0 e^x\, dx = \lim_a \int_{-a}^0 e^x\, dx = \lim_a (1-e^{-a}) = 1$

(c) $\int_1^\infty \frac{\log x}{x^2}\, dx = \lim_{a\to\infty} \int_1^a \frac{\log x}{x^2}\, dx$

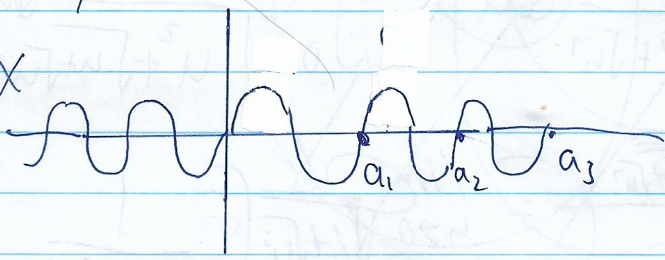
$\int_1^a \frac{\log x}{x^2}\, dx = -\int_1^a \log x \, d\frac{1}{x} = -\frac{\log x}{x}\Big|_1^a + \int_1^a \frac{1}{x^2}\, dx$

$= -\frac{\log a}{a} + 0 - \frac{1}{x}\Big|_1^a$

$= \frac{1+\log a}{a} + 1$

Take $a \to \infty$, $\int_1^\infty \frac{\log x}{x^2}\, dx = 1$

(c) $\int_{-\infty}^{\infty} \sin(x)\, dx$



$\int_0^a \sin x = -\cos a + 1$

$a_n = 2\pi n$ $\quad \lim_{n\to\infty} \int_0^{a_n} \sin x\, dx = 0$

$b_n = 2\pi n + \frac{\pi}{2}$ $\quad \lim_{n\to\infty} \int_0^{b_n} \sin x\, dx = 1$

(d) $\int_1^\infty \sin x^2\, dx$

$\int_0^a \sin(x^2)\, dx = \int_0^{\frac{\sqrt{a}}{2}} \frac{\sin 2u}{\sqrt{2}\sqrt{u}}\, du$ (Let $2u = x^2$)

$$= \int_0^{\sqrt{a/2}} \frac{1}{\sqrt{2}\sqrt{u}}\, d\sin^2 u = \frac{1}{\sqrt{2}}\left[\frac{\sin^2 u}{\sqrt{u}}\right]_1^{\sqrt{a/2}} + \frac{1}{2\sqrt{2}}\int_1^{\sqrt{a/2}} \frac{\sin^2 u}{u^{3/2}}\, du$$

$$\lim_{a\to\infty} \int_0^{\sqrt{a/2}} \frac{1}{\sqrt{2}\sqrt{u}}\, d\sin^2 u \le -\frac{1}{\sqrt{2}}\sin^2 1 + \int_1^{\infty} \frac{1}{u^{3/2}}\, du < \infty$$

Hence $\int_0^{\infty} \sin(x^2)\, dx$ converges

(f) Shown in lecture notes

(g) $\int_0^1 \frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\, dx \le \int_0^1 \frac{1}{\sqrt{x}}$,

(h) $\int_1^{\infty} \frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\, dx$ Let $u = 1/x$, $du = -\frac{1}{x^2}\, dx$, $-\frac{1}{u^2}\, du = dx$

$$= -\int_1^0 \frac{1}{u^2\sqrt{\frac{1}{u}+\sqrt{\frac{1}{u}+\sqrt{\frac{1}{u}}}}}\, du$$

$$= \int_0^1 \frac{1}{\sqrt{u^3+\sqrt{u^5+\sqrt{u^7}}}}\, du$$

$$\lim_{u\to 0} \frac{\frac{1}{\sqrt{u^3+\sqrt{u^5+\sqrt{u^7}}}}}{\frac{1}{\sqrt{u^3}}} = \lim_{u\to 0} \frac{1}{\sqrt{1+\sqrt{u}}} \ldots$$ [illegible]

(h) $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x+\sqrt{x+\sqrt{x}}}}\,dx$

$$\geq \int_1^{\infty} \frac{1}{\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}}\,dx$$

let $y = \sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}$

$y = \sqrt{x+y}$

$y^2 - y = x \qquad dx = (2y-1)\,dy$

$y = \frac{1}{2} + \frac{\sqrt{4x+1}}{2}$

$$= \int_{\frac{1}{2}+\frac{\sqrt{5}}{2}}^{\infty} \frac{1}{y}(2y-1)\,dy$$

$$= \int_{\frac{1}{2}+\frac{\sqrt{5}}{2}}^{\infty} 2 - \frac{1}{y}\,dy$$

$$= \lim_{a\to\infty}\left(2y - \ln y \Big|_{\frac{1}{2}+\frac{\sqrt{5}}{2}}^{a}\right)$$

$$\geq \lim_{a\to\infty} y \Big|_{\frac{1}{2}+\frac{\sqrt{5}}{2}}^{a}$$

$$= \infty$$

Hence it diverges