香港科技大學
THE HONG KONG
UNIVERSITY OF SCIENCE
AND TECHNOLOGY

數學系
DEPARTMENT OF MATHEMATICS

MIDTERM EXAMINATION

Course Code: MATH 1024
Course Title: Honors Calculus II
Semester: Spring 2020-21
Date and Time: 2:00PM - 5:00PM, 20 March 2021

Instructions

- Absolutely **NO DISCUSSION**, online or offline, with any person other than the instructor.
- Posting anything on social media or any public internet website is **NOT** allowed.
- It is an **OPEN-NOTES** and **OPEN-INTERNET** exam. You can do searching on the internet (including using WolframAlpha), but no posting and discussion.
- Only results discussed in lectures and tutorials, and results proved in homework can be directly quoted.
- You must **SHOW YOUR WORK** and **JUSTIFY YOUR STEPS** to receive credits in every problem in Part B.
- Some problems in Part B are structured into several parts. You can quote the results stated in the preceding parts to do the next part, even if you cannot do the preceding parts.
- One page 1 of your work, write the following statement and sign:

  > "I confirm that I have answered the questions using only materials specified approved for use in this examination, that all the answers are my own work, and that I have not received any assistance during the examination."

  Your signature

- You need to **SIGN** on the top right corner of **EVERY PAGE** of your work.

HKUST Academic Honor Code

Honesty and integrity are central to the academic work of HKUST. Students of the University must observe and uphold the highest standards of academic integrity and honesty in all the work they do throughout their program of study. As members of the University community, students have the responsibility to help maintain the academic reputation of HKUST in its academic endeavors. Sanctions will be imposed on students, if they are found to have violated the regulations governing academic integrity and honesty.

Part A - Short Questions (25 points)

Recommended timing: the whole Part A $<$ 30min.
Instruction: Write down your answers or solutions in your own answer sheets. **State the question and part numbers clearly**.

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1. What is the name of the problem about finding the exact value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$? **[2]**
   A. Bessel Problem
   B. Bravo Problem
   C. Biggest Problem
   D. Basel Problem
   E. Batman Problem

2. Given that X and Y are bounded subsets in $\mathbb{R}^2$ and $X \subset Y$. **[6]**

   In each ______ below, fill in one of the symbols: $\subset$, $\supset$, $\leq$, $\geq$:

   (a) $\mu_*(X)$ ______ $A(S)$ where $S \subset X$, S is a simple region
   (b) $\{A(T) : T \supset X,\ T \text{ is a simple region}\}$ ______ $\{A(T) : T \supset Y,\ T \text{ is a simple region}\}$
   (c) $\mu^*(X)$ ______ $\mu^*(Y)$
   (d) $\mu_*(X)$ ______ $\mu_*(Y)$
   (e) $\mu_*(X)$ ______ $\mu^*(X)$
   (f) $\mathbb{R}^2 \backslash X$ ______ $\mathbb{R}^2 \backslash Y$

3. Which of the following types of functions must be Riemann integrable over $[a, b]$? List ALL correct answer(s). **[5]**
   A. continuous functions on $[a, b]$
   B. bounded functions on $[a, b]$
   C. strictly decreasing functions on $[a, b]$
   D. positive functions on $[a, b]$
   E. constant functions

4. Consider the function $f : [0, 1] \to \mathbb{R}$ defined as:

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   Let P_n be the uniform partition of $[0, 1]$.

   (a) Find the exact value of **[5]**

$$\lim_{n\to\infty} \big(U(f, P_n) - L(f, P_n)\big).$$

   (b) Based on the knowledge you have learned in this course, is the above result sufficient to show that f is not Riemann integrable on $[0, 1]$? Explain briefly. **[2]**

5. Determine whether the following improper integral converges or diverges: **[5]**

$$\int_{-\infty}^{+\infty} \frac{1}{x^3 - 1}\, dx.$$

Part B - Long Questions (75 points): Answer ALL THREE problems

Recommended timing: Q1 $<$ 40min, Q2 $<$ 50min, Q3 $<$ 60min
Instructions: Write your solutions on your own answer sheets. **Clearly indicate the question and part numbers.**

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1. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of positive numbers such that $\lim_{n\to\infty} a_n = 1$ (note that $\{a_n\}$ is NOT given to be monotone). Suppose $f : [0,1] \to \mathbb{R}$ is a continuous function such that:
   - $f(0) = 1$,
   - for any $n \in \mathbb{N} \cup \{0\}$, $f(\frac{1}{2^n}) = a_n$, and
   - for any $n \in \mathbb{N} \cup \{0\}$, we have $f''(x) = 0$ for any $x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n})$.

   (a) Write down the explicit expression (in terms of a_n's) of the 2^n-th trapezoidal sum T_{2^n} of f over $[0,1]$ using the uniform partition P_{2^n} with 2^n sub-intervals. Explain briefly your answer. Use diagram(s) if needed. **[10]**

   (b) i. Find the explicit expression of $U(f, P_{2^n}) - L(f, P_{2^n})$ in terms of a_n's. Explain briefly your answer. Use diagram(s) if needed. **[10]**

   ii. Hence, show that f is Riemann integrable on $[0,1]$. **[5]**

   [Attention: Your final expressions in both (b) and (c)i should NOT involve any "f".]

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2. **Remark: You can use without proof any relevant results proved in class.**

   For each $n \in \mathbb{N} \cup \{0\}$, we define:
$$A_n := \int_0^{\pi/2} \cos^{2n} x\, dx, \quad B_n := \int_0^{\pi/2} x^2 \cos^{2n} x\, dx, \text{ and } \quad C_n := \int_0^{\pi/2} x^4 \cos^{2n} x\, dx.$$

   (a) Prove that for any $n \in \mathbb{N}$, we have **[10]**
$$\frac{C_{n-1}}{A_{n-1}} - \frac{C_n}{A_n} = \frac{3}{n^2}\frac{B_n}{A_n}.$$

   (b) Prove that there exists a constant $K > 0$ such that $\frac{C_n}{A_n} \le \frac{K}{n}$ for any $n \in \mathbb{N}$. **[4]**

   (c) It is known that $\zeta(4) := \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. Using this fact, results from (a), (b), and related results proved in class, show that **[10]**
$$\frac{1}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{n^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}\right) \in \mathbb{Q}.$$

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3. Let $x_0 < x_1 < \cdots < x_n$ be $(n+1)$ distinct points on $\mathbb{R}$. Define:
$$P(x) := \sum_{i=0}^{n} e^{x_i^2} \cdot \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)},$$
$$Q(x) := (x-x_0)(x-x_1)\cdots(x-x_n).$$

   For each $t \in \mathbb{R}\backslash\{x_0, x_1, \cdots, x_n\}$, we define
$$G_t(x) := Q(t)e^{x^2} - Q(x)e^{t^2} - Q(t)P(x) + Q(x)P(t).$$

(a) Show that for each $t \in \mathbb{R}\backslash\{x_0, x_1, \cdots, x_n\}$, there exists $\xi_t \in \mathbb{R}$, which may depend on t, such that $G_t^{(n+1)}(\xi_t) = 0$. Here $G_t^{(n+1)}$ denotes the $(n+1)$-th derivative with respect to x. [Hint: Consider $G_t(x_i)$'s.] **[8]**

(b) Let ξ_t be as obtained in (a). Show that **[8]**

$$e^{t^2} = P(t) + \frac{Q(t)h(\xi_t)e^{\xi_t^2}}{(n+1)!} \quad \forall t \in \mathbb{R}\backslash\{x_0, x_1, \cdots, x_n\}$$

where $h(x)$ is a polynomial with coefficients in $[0, \infty)$.

(c) Assume further that $x_0 = 0$ and $x_n = 2$. Show that **[8]**

$$\left| \int_0^2 e^{t^2}\, dt - \int_0^2 P(t)\, dt \right| \leq \frac{2^{n+2}e^4 h(2)}{(n+1)!}.$$

(d) Do you agree with the following argument? Explain why or why not. **[2]**

As $\displaystyle\lim_{n\to\infty} \frac{2^{n+2}}{(n+1)!} = 0$, letting $n \to \infty$ on the result in (c), we conclude that

$$\int_0^2 e^{t^2}\, dt = \int_0^2 P(t)\, dt.$$

- End of Paper -