$f : [a, +\infty) \longrightarrow \mathbb{R}$ $\qquad \int_a^{+\infty} f(x)\,dx, \quad \int_{-\infty}^{a} f(x)\,dx \;. \quad \int_{-\infty}^{\infty} f(x)\,dx$

$(-\infty, a]$

$(-\infty, \infty)$

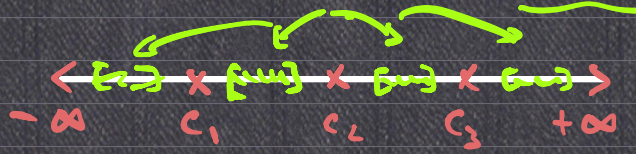
$f : [a, b) \longrightarrow \mathbb{R}. \qquad \int_a^b f(x)\,dx$

$\lim_{x \to c} |f(x)| = \infty.$

$\quad$ in $[a,b]$

f is Riemann integrable on each $[a,b] \not\ni c_1, c_2, c_3$

(and f bounded)



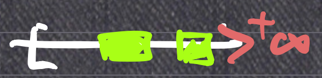
$$\int_{-\infty}^{\infty} f(x)\,dx = \left(\int_{-\infty}^{c_1} + \int_{c_1}^{c_2} + \int_{c_2}^{c_3} + \int_{c_3}^{+\infty} \right) f(x)\,dx$$

$$\int_{c_1}^{c_2} f(x)\,dx = \lim_{\alpha \to c_1^+} \int_{\alpha}^{\frac{c_1+c_2}{2}} f(x)\,dx + \lim_{\beta \to c_2^-} \int_{\frac{c_1+c_2}{2}}^{\beta} f(x)\,dx$$

$$\Gamma(x) := \int_0^{+\infty} t^{x-1} e^{-t}\,dt$$

Gamma

Prop: Given $\begin{cases} f, g \text{ locally Riemann integrable on } [a, +\infty) \\ 0 \le f(x) \le g(x) \text{ on } [N, +\infty) \\ \qquad \exists N > a. \end{cases}$



- If $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges.

Proof: $F(x) := \int_N^x \underbrace{f(t)}_{\ge 0}\,dt, \qquad G(x) := \int_N^x \underbrace{g(t)}_{\ge 0}\,dt.$

$F(x)$, $G(x)$ are monotone increasing.

$\left[y > x, \quad F(y) - F(x) = \int_x^y \underbrace{f(t)}_{\oplus} dt \geq 0. \right]$
$(y > x)$

$$\mathbb{R} \ni \underbrace{\int_N^{+\infty} g(t)\,dt}_{\text{converges}} = \lim_{x \to +\infty} G(x) = \sup_{[N,+\infty)} G(x) \Rightarrow G(x) \text{ is bounded on } [N, +\infty).$$

$$f(t) \leq g(t) \text{ on } t \in [N, +\infty) \Rightarrow F(x) = \int_N^x f(t)\,dt \leq \int_N^x g(t)\,dt = G(x)$$

on $[N, +\infty)$.

(G bounded)
$\Rightarrow$ $F(x)$ is bounded from above.

$\therefore \lim_{x \to +\infty} F(x)$ exists. $\Rightarrow \int_N^{+\infty} f(t)\,dt$ converges.

$$\Rightarrow \int_a^{+\infty} f(t)\,dt = \int_a^N f(t)\,dt + \int_N^{+\infty} f(t)\,dt$$

converges.

$\boxed{\Xi}$.

Ex.

$$\int_2^{+\infty} \frac{1}{\sqrt[3]{x^2 - 1}}\,dx$$

$$\sqrt[3]{x^2-1} \leq x^{2/3}$$

$$\frac{1}{\sqrt[3]{x^2-1}} \geq \frac{1}{x^{2/3}}$$

$\int_2^{+\infty} \frac{1}{x^{2/3}}\,dx$ diverges. $2/3 < 1$

$\left(\frac{2}{3} < 1\right)$ $\therefore$

$$\Rightarrow \int_2^{+\infty} \frac{1}{\sqrt[3]{x^2-1}} \text{ diverges}$$

$$\int_0^1 \frac{1}{\sqrt[3]{x^2+2x^4}}\,dx$$

$$x^2+2x^4 \geq x^2$$

$$0 \leq \frac{1}{\sqrt[3]{x^2+2x^4}} \leq \frac{1}{x^{2/3}}$$

$\int_0^1 \frac{1}{x^{2/3}}\,dx$ converges

$\Rightarrow \int_0^1 \frac{1}{\sqrt[3]{x^2+2x^4}}\,dx$ converges.

$$\sqrt[3]{x^2-2x^4}$$

$$\geq \sqrt[3]{\frac{1}{2}x^2} = \frac{1}{\sqrt[3]{2}}x^{\frac{2}{3}}$$

on $x \in [0, \frac{1}{2}]$.

$$x^2-2x^4 \geq \frac{1}{2}x^2$$

$$\Leftrightarrow \frac{1}{2}x^2 \geq 2x^4$$

$$\Leftrightarrow x^2 \leq \frac{1}{4}$$

$$\frac{1}{\sqrt[3]{x^2-2x^4}} \leq \frac{1}{\sqrt[3]{2}}\frac{1}{x^{2/3}}$$

Prop: (Limit Comparison Test)

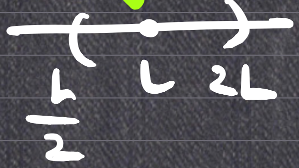
f, g locally Riem. integrable on $[a, +\infty)$, $0 \leq f(x), g(x)$.

Given $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L$

- If $L \in (0, \infty)$, $\int_a^{+\infty} f(x)\,dx$ converges iff $\int_a^{+\infty} g(x)\,dx$ converges.
- If $L = 0$, $\int_a^{+\infty} g(x)\,dx$ converges $\Rightarrow \int_a^{+\infty} f(x)\,dx$ converges.

• If $L = +\infty$, $\int_a^{+\infty} f(x)\,dx$ converges $\Rightarrow$ $\int_a^{+\infty} g(x)\,dx$ converges

<u>Proof:</u> If $L \in (0, +\infty)$, $\dfrac{f(x)}{g(x)} \to L$



$\exists N > 0$ s.t.

$$\frac{L}{2} \le \frac{f(x)}{g(x)} \le 2L \quad \forall x \in [N, +\infty).$$

$$\Rightarrow \underline{\frac{L}{2} g(x) \le f(x) \le 2L\, g(x)}$$

If $L = 0$, $\dfrac{f(x)}{g(x)} \to 0$ as $x \to +\infty$.

$\exists N > 0$ s.t.

$$0 \le \frac{f(x)}{g(x)} \le 1 \quad \text{on } [N, +\infty)$$

$$\underline{0 \le f(x) \le g(x)}$$

∎

<u>e.g.</u> $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5x \pm 2x^4}}\,dx$ $\qquad \dfrac{1}{\sqrt[3]{5x \pm 2x^4}} \approx \dfrac{1}{\sqrt[3]{5x}}$ when $x \approx 0$.

$$\lim_{x \to 0^+} \frac{\frac{1}{\sqrt[3]{5x \pm 2x^4}}}{\frac{1}{\sqrt[3]{x}}} = \lim_{x \to 0^+} \sqrt[3]{\frac{x}{5x \pm 2x^4}} = \lim_{x \to 0^+} \sqrt[3]{\frac{1}{5 \pm 2x^3}}$$

$$= \frac{1}{5^{1/3}} \in (0, +\infty).$$

$\displaystyle\int_0^5 \frac{1}{\sqrt[3]{x}}\,dx$ converges $\Longrightarrow$ $\displaystyle\int_0^5 \frac{1}{\sqrt[3]{5x \pm 2x^4}}\,dx$ converges.

⚠

$$-\frac{1}{x} \leq \frac{1}{x^2} \quad \forall x \in [1,+\infty).$$

$\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges, but $\int_1^{+\infty} -\frac{1}{x}\,dx$ diverges.

<u>Prop:</u> (Absolute converge test)

$$\int_a^{+\infty} |f(x)|\,dx \text{ converges} \Longrightarrow \int_a^{+\infty} f(x)\,dx \text{ converges.}$$

<u>e.g.</u> $$\int_3^{+\infty} \frac{\sin x}{(x-1)(x-2)}\,dx$$

Consider first $\int_3^{+\infty} \left|\frac{\sin x}{(x-1)(x-2)}\right| dx$.

(*) $0 \leq \left|\frac{\sin x}{(x-1)(x-2)}\right| \leq \frac{1}{(x-1)(x-2)}$ on $[3,+\infty)$

$$\left\{\begin{array}{l} \lim_{x\to+\infty} \frac{\frac{1}{(x-1)(x-2)}}{\frac{1}{x^2}} = 1 \in (0,+\infty) \\ \int_3^{+\infty} \frac{1}{x^2}\,dx \text{ converges} \\ (2>1) \end{array}\right. \underset{\text{limit comparison test}}{\Longrightarrow} \int_3^{+\infty} \frac{1}{(x-1)(x-2)}\,dx \text{ converges.}$$

(*) $\Rightarrow$ $\int_3^{+\infty} \left|\frac{\sin x}{(x-1)(x-2)}\right| dx$ converges.

$\Rightarrow \int_3^{+\infty} \frac{\sin x}{(x-1)(x-2)}\,dx$ converges.

Proof of Abs. Conv. Test :

$$0 \le f(x) + |f(x)| \le 2|f(x)| \quad (\star).$$

$\int_a^{+\infty} |f(x)|\,dx$ converges $\Rightarrow$ $\int_a^{+\infty} (f(x) + |f(x)|)\,dx$ converges.

$$\int_a^{+\infty} f(x)\,dx = \int_a^{+\infty} (f(x) + |f(x)|) - |f(x)|\,dx$$

$$= \underbrace{\int_a^{+\infty} (f(x) + |f(x)|)\,dx}_{\text{conv.}} - \underbrace{\int_a^{+\infty} |f(x)|\,dx}_{\text{conv.}}$$

$\square$

e.g.

$\int_1^{+\infty} \frac{\sin x}{x}\,dx$ converges :

Proof:

$$\int_1^{+\infty} \frac{\sin x}{x}\,dx = \lim_{a\to+\infty} \int_1^a \frac{\sin x}{x}\,dx$$

$$= \lim_{a\to+\infty} \int_1^a \frac{1}{x}\,d(-\cos x)$$

$$= \lim_{a\to+\infty} \left(\left[-\frac{\cos x}{x} \right]_1^a - \int_1^a (-\cos x)\cdot\left(-\frac{1}{x^2}\right) dx \right)$$

$$= \lim_{a\to+\infty} \left(\underbrace{-\frac{\cos a}{a}}_{\to\, 0} + \underbrace{\cos 1}_{\checkmark} - \underbrace{\int_1^a \frac{\cos x}{x^2}\,dx}_{?} \right)$$

exists

$0 \le \left|\frac{\cos x}{x^2}\right| \le \frac{1}{x^2}$ on $[1, +\infty)$.

$\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges $\Rightarrow$ $\int_1^{+\infty} \left|\frac{\cos x}{x^2}\right| dx$ converges

$\Rightarrow \int_1^{+\infty} \frac{\cos x}{x^2}\,dx$ converges.

$\therefore \int_1^{+\infty} \frac{\sin x}{x}\,dx$ converges

But: $\int_\pi^{+\infty} \left|\frac{\sin x}{x}\right| dx$

$y = |\sin x|$

$\pi \le x \le 2\pi$

$\frac{1}{2\pi} \le \frac{1}{x} \le \frac{1}{\pi}$ $\quad \frac{1}{x} \ge \frac{1}{3\pi}$

$$\ge \int_\pi^{2\pi} \left|\frac{\sin x}{x}\right| dx + \int_\pi^{2\pi} \left|\frac{\sin x}{x}\right| dx + \int_{2\pi}^{3\pi} \left|\frac{\sin x}{x}\right| dx + \cdots$$

$$\ge \int_\pi^{2\pi} \frac{|\sin x|}{2\pi}\,dx + \int_{2\pi}^{3\pi} \frac{|\sin x|}{3\pi}\,dx + \int_{3\pi}^{4\pi} \frac{|\sin x|}{4\pi}\,dx + \cdots$$

$$= \underbrace{\left(\frac{1}{2\pi} + \frac{1}{3\pi} + \frac{1}{4\pi} + \cdots\right)}_{+\infty.} \int_0^\pi |\sin x|\,dx$$

$\int_\pi^{+\infty} \left|\frac{\sin x}{x}\right| dx$ diverges.

$$\int_a^{+\infty} f(x)\,g(x)\,dx = \int_a^{+\infty} g(x)\,d\underbrace{F(x)}$$

$$F(x) = \int_a^x f(t)\,dt$$

(compare with $f = \sin x$, $g = \frac{1}{x}$)

$$= \lim_{b\to+\infty}\left(\Big[g(x)F(x)\Big]_a^b - \int_a^b F(x)\,g'(x)\,dx\right)$$

$$= \lim_{b\to+\infty}\Big(g(b)F(b) - \underbrace{g(a)F(a)}_{\text{const.}} - \int_a^b F(x)\,g'(x)\,dx\Big)$$

WANT: $\begin{cases} \lim\limits_{b\to+\infty} g(b)F(b) \text{ exists} \\ \int_a^{+\infty} F(x)\,g'(x)\,dx \text{ converges} \end{cases}$

$\Rightarrow \int_a^{+\infty} f(x)\,g(x)\,dx$ converges.

Dirichlet Test: $\begin{cases} F(x) \text{ bounded} \\ g \text{ monotone and } \lim\limits_{x\to+\infty} g(x) = 0. \\ \;\; C^1 \end{cases}$ ✓

Abel Test: $\begin{cases} \lim\limits_{x\to+\infty} F(x) \text{ exists,} \\ g \text{ is } C^1\text{, monotone, bounded.} \end{cases}$